An Introduction to Graph Rewriting

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Graph Rewriting: Motivation

Handling real-world data structures

A Circular Linked List

A Doubly-Linked Circular List
Graph Rewriting: Motivation

Efficient Implementations
Graph Rewriting: Motivation

Various Application Domains

Programming, Graph Grammars, UML-like Modeling, Databases, etc.
Graph Rewriting

Various Definitions of Graphs

- Undirected graphs
- Directed graphs
- Labeled graphs
- Hypergraphs
- Multigraphs
- Rooted graphs
- Attributed graphs
- ...

\[ n_0 \rightarrow n_1 \rightarrow n_3 \rightarrow n_4 \rightarrow n_5 \]

\[ n_0 \rightarrow n_1 \rightarrow n_3 \rightarrow n_4 \rightarrow n_5 \]

\[ n_0 : l_0 \rightarrow n_1 : l_1 \rightarrow n_3 : l_3 \rightarrow n_4 : l_4 \rightarrow n_5 : l_5 \]

\[ R_0 \rightarrow R_1 \rightarrow R_3 \rightarrow R_7 \]

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Graph Rewriting

\[\text{Diagram: Graph transformation rules.}\]
Graph Rewriting
Graph Rewriting : Elementary Actions

There are different possible elementary actions on graphs.

- **Delete** an existing item (node or edge)
- **Add** a new item
- **Merge** two or more items
- **Clone** (copy) an item or a subgraph
- ...

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Graph Rewriting: Elementary Actions

- Delete
- Add
- Merge
- Clone
Since late 1960’s!

- There are several approaches, in the literature, to rewrite graphs:
  - Imperative Programs
  - Rule-Based Programs
  - Graph Grammars
  - Knowledge-Base updates
  - Non-classical Logics
  - ...
Graph Rewriting

Different frameworks

Since late 1960’s!

- There are several approaches, in the literature, to rewrite graphs:
  - Imperative Programs
  - Rule-Based Programs
    - Algebraic/Categorial approaches (DPO, SPO, SqPO, PBPO, . . .)
    - Algorithmic approaches
  - Graph Grammars
  - Knowledge-Base updates
  - Non-classical Logics
  - ...
Some References

- Fundamentals of Algebraic Graph Transformation
  - ISR2019 (Paris) July 1 and 2, 2019
  - R. Echahed

- Handbook of Graph Grammars and Computing by Graph Transformation
  - Volume 1: Foundations
  - Volume 2: Applications, Languages and Tools
  - Volume 3: Concurrency, Parallelism, and Distribution


- World Scientific Publishing Company
Outline

1. Introduction
2. Preliminary Definitions
3. Graph Rewriting: Elementary Actions
4. Some Algebraic Approaches to Graph Rewriting
5. Attributed Graph Transformation and PBPO rules
6. Termgraph Rewriting: An Algorithmic Approach
7. Verification of Graph Transformation
Categories

A category $C=\langle Obj_C, Hom_C, \circ, id \rangle$ consists of

- A class $Obj_C$ of objects
- A class $Hom_C$ of morphisms. We write $Hom_C(A, B)$ for the morphisms from object $A$ to $B$ and $f : A \rightarrow B$ an element of $Hom_C(A, B)$
- A composition of morphisms $\circ$. For all objects, A,B and C, $\circ : Hom_C(A, B) \times Hom_C(B, C) \rightarrow Hom_C(A, C)$.

Such that:

- The composition $\circ$ is associative: For all morphisms $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$, $(h \circ g) \circ f = h \circ (g \circ f)$ and
- For every object $A$, there exists a morphism $id_A : A \rightarrow A$ called the identity such that: for all morphism $f : A \rightarrow B$, $f \circ id_A = f$ and $id_B \circ f = f$. 
Examples of categories

Category of sets:
- objects are sets
- morphisms are functions

Category of graphs:
- objects are graphs
- morphisms are graph homomorphisms
Graphs

In this talk we consider the category of graphs where objects and morphisms are defined as follows:

A graph (or multigraph) $G = (N_G, E_G, s_G, t_G)$ consists of

- a set of nodes $N_G$
- a set of edges $E_G$
- a source function $s_G : E_G \rightarrow N_G$
- a target function $t_G : E_G \rightarrow N_G$

A graph homomorphism between two graphs $G$ and $T$, $h : G \rightarrow T$, consists of two functions $h_N : N_G \rightarrow N_T$ and $h_E : E_G \rightarrow E_T$ such that:

- $h_N \circ s_G = s_T \circ h_E$
- $h_N \circ t_G = t_T \circ h_E$
Graph Homomorphism: Example

Graph $G$  

$N_G = \{f, a, b\}$ and $E_G = \{e_1, e_2\}$

Graph $T$  

$N_T = \{g, c\}$ and $E_T = \{f_1, f_2, f_3\}$

Notice that symbols $f, a, b, c, g$ represent nodes and not function symbols!

A first homomorphism $h : G \rightarrow T$ can be defined as follows:

$h_N(f) = g$ and $h_N(a) = h_N(b) = c$

$h_E(e_1) = f_1$ and $h_E(e_2) = f_2$
Graph Homomorphism: Example

Graph G

$$N_G = \{f, a, b\} \text{ and } E_G = \{e_1, e_2\}$$

Graph T

$$N_T = \{g, c\} \text{ and } E_T = \{f_1, f_2, f_3\}$$

Notice that symbols $f, a, b, c, g$ represent nodes and not function symbols!

A second homomorphism $k : G \rightarrow T$ can be defined as follows:

$$k_N(f) = k_N(a) = k_N(b) = g$$
$$k_E(e_1) = k_E(e_2) = f_3$$

Are there other homomorphisms between $G$ and $T$?
Pushout

Definition

The **Pushout** of morphisms $f$ and $g$ consists of an object $D$ and two morphisms $f'$ and $g'$ such that:

- **Commutativity**
  \[ g' \circ f = f' \circ g, \text{ and} \]

- **Universal Property**
  For all objects $D'$ and morphisms $u$ and $v$ such that $u \circ f = v \circ g$, there exists a unique morphism $h : D \to D'$ such that $h \circ g' = u$ and $h \circ f' = v$. 

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow g & \downarrow g' \\
C \xrightarrow{f'} D \\
\downarrow u & \downarrow h \\
D' \xrightarrow{v} D'
\end{array}
\]
Pushout

In Sets:

- \( D = (B + C)/\equiv \)
  - with \( \equiv \) being the least equivalence generated by the pairs \( \{(f(x), g(x)) \mid x \in A\} \) over \( B + C \).
- For all \( x \in B, g'(x) = \bar{x} \)
- For all \( x \in C, f'(x) = \bar{x} \)
Pushout: Example 1
Pushout: Example 2

\[ f(1) = a, \quad f(2) = b, \quad f(3) = b, \quad f(4) = c \]
\[ g(1) = f, \quad g(2) = f, \quad g(3) = e, \quad g(4) = d \]
Pushout: Example 2

\[ f(1) = a, f(2) = b, f(3) = b, f(4) = c \]
\[ g(1) = f, g(2) = f, g(3) = e, g(4) = d \]
**Pushout: Example 3**

In graphs: The sets of nodes and edges of the pushout object \((D)\) can be constructed componentwise as pushouts in \(\text{Sets}\) (respecting the source and target functions).

\[
f(1) = a, f(2) = a, f(3) = b, f(e_1) = f(e_2) = e_3\]

\[
g(1) = n, g(2) = d, g(3) = n, g(e_1) = e_5, g(e_2) = e_4\]
Pushout: Example 3

\[
\begin{align*}
&\begin{array}{c}
1 \\
e_1 \\
\downarrow \\
e_4
\end{array} \quad \begin{array}{c}
2 \\
e_2 \\
\downarrow \\
e_5
\end{array} \\
\Rightarrow \begin{array}{c}
a \\
\downarrow e_3 \\
b
\end{array} \\
\begin{array}{c}
d \\
e_4 \\
\downarrow \\
e_5
\end{array} \quad \begin{array}{c}
n \\
e_5 \\
\downarrow \\
\bar{a} \\
\bar{e}_3
\end{array}
\end{align*}
\]

\[
f(1) = a, f(2) = a, f(3) = b, f(e_1) = f(e_2) = e_3 \\
g(1) = n, g(2) = d, g(3) = n, g(e_1) = e_5, g(e_2) = e_4
\]

In graphs: The sets of nodes and edges of the pushout object \((D)\) can be constructed componentwise as pushouts in Sets (respecting the source and target functions)
The *Pullback* of morphisms $f$ and $g$ consists of an object $D$ and two morphisms $f'$ and $g'$ such that:

- **Commutativity**
  $$f \circ g' = g \circ f',$$
  and

- **Universal Property**
  For all objects $D'$ and morphisms $u$ and $v$ such that $f \circ u = g \circ v$, there exists a unique morphism $h : D' \to D$ such that $g' \circ h = u$ and $f' \circ h = v$. 

In Sets,

- \( D = \{(x, y) \in B \times C \mid f(x) = g(y)\} \)
- For all \((b, c) \in D, g'(b, c) = b\)
- For all \((b, c) \in D, f'(b, c) = c\)
Pullback
**Pullback**

In graphs: The sets of nodes and edges of pullback object $D$ can be constructed componentwise as pullbacks in $\text{Sets}$ (respecting the source and target functions).

$f(1) = a, f(2) = b, f(e_1) = f(e_2) = e_3$

$g(3) = a, g(4) = g(5) = b, g(e_4) = g(e_5) = e_3$
Pullback

In graphs: The sets of nodes and edges of pullback object $D$ can be constructed componentwise as pullbacks in Sets (respecting the source and target functions)

$$f(1) = a, f(2) = b, f(e_1) = f(e_2) = e_3$$
$$g(3) = a, g(4) = g(5) = b, g(e_4) = g(e_5) = e_3$$
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Adding New Items

Pushouts can be used to add new items to a graph.

![Diagram showing the process of adding new items to a graph using pushouts.](image-url)
Merging Existing Items

Pushouts can be used to merge existing items of a graph.
Deleting Existing Items

- Both Pushouts and Pullbacks can be used to delete items within a graph!
- Single pushout can be used to delete items in a graph but requires partial morphisms (out of this talk).

Use of pushout complement: A **pushout complement (POC)** of two morphisms \( m : L \to G \) and \( l : K \to L \) is an object \( D \) and two morphisms \( l' : D \to G \) and \( m' : K \to D \) such that the following diagram is a pushout:

\[
\begin{array}{cc}
L & \xleftarrow{l} & K \\
\downarrow{m} & & \downarrow{m'} \\
G & \xleftarrow{l'} & D
\end{array}
\]
Deleting Existing Items

Example of the use of pushout complement

```
        f   f
       ↓  ↓
      a   a
        ↓  ↓
      r   r
     ↓  ↓
    f   f
   ↓  ↓
  a   a
```

Remark: Pushout complements may not exist or not be unique!
Pushout Complement

Pushout complements may not be unique!

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a}
\end{array}
\hspace{2cm}
\begin{array}{c}
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a} \\
\downarrow \\
\text{a}'
\end{array}
\end{array}
\end{array}
\hspace{2cm}
\begin{array}{c}
\begin{array}{c}
\text{r} \\
\downarrow \\
\text{f} \\
\downarrow \\
\text{a}
\end{array}
\hspace{2cm}
\begin{array}{c}
\begin{array}{c}
\text{r} \\
\downarrow \\
\text{f}
\end{array}
\end{array}
\end{array}
\end{array}
\]
Pushout Complement

Pushout complements may not be unique!
Pushout Complement

Exercise
Pushout Complement

Pushout complement may not exist!

```
\[
\begin{array}{c}
f \\
\downarrow \\
a \\
\downarrow \\
r \\
\downarrow \\
f \\
\downarrow \\
a \\
\end{array}
\quad \quad \quad \quad \quad \quad \quad \\
\begin{array}{ccc}
f & \quad & \text{no POC}
\end{array}
\]```
Pushout Complement

Pushout complement may not exist!

\[
\begin{array}{c}
\begin{array}{c}
\text{no POC}
\end{array}
\end{array}
\]
Pushout Complement

\[
\begin{array}{c}
a \\
a' \\
\downarrow f \\
a
\end{array}
\quad\quad
\begin{array}{c}
a \\
\downarrow \\
POC?
\end{array}
\]
Pushout Complement

Pushout complement may not exist!

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
a' \\
f \\
a
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
\text{no POC}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]
Pushout Complement

Exercise

\[
\begin{array}{c}
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a}
\end{array}
\end{array}
\quad \leftarrow \quad
\begin{array}{c}
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a}
\end{array}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a}
\end{array}
\end{array}
\quad \leftarrow \quad
\begin{array}{c}
\begin{array}{c}
\text{POC?}
\end{array}
\end{array}
\end{array}
\]
Pushout Complement

Pushout complement may not exist!

\[
\begin{array}{ccc}
 f & \leftarrow & f \\
a & \downarrow & a \\
\end{array}
\]

\[
\begin{array}{ccc}
 f & \leftarrow & f \\
a & \downarrow & a \\
\end{array}
\]

no POC
Existence of Pushout Complements (in Graphs)

Let \( m : L \to G \) and \( l : K \to L \) be two graph morphisms. There exists a pushout complement defined by a graph \( D \) and two morphisms \( l' : D \to G \) and \( m' : K \to D \) iff the following gluing conditions hold:

- **Dangling Condition:**
  \[
  \{ n \in N_L \mid \exists e \in E_G \setminus m(E_L), s_G(e) = m(n) \text{ or } t_G(e) = m(n) \} \subseteq l(N_K)
  \]

- **Identification Condition:**
  \[
  \begin{align*}
  &\{ n \in N_L \mid \exists n' \in N_L, n \neq n' \text{ and } m(n) = m(n') \} \subseteq l(N_K) \\
  &\{ e \in E_L \mid \exists e' \in E_L, e \neq e' \text{ and } m(e) = m(e') \} \subseteq l(E_K)
  \end{align*}
  \]
Deleting Existing Items

Use of pullbacks: Example
Deleting Existing Items

Use of pullbacks: Example
Deleting Existing Items

Use of pullbacks: Example
Cloning Items
Use of Pullbacks

Cloning the subgraph containing nodes $l_0, l_1, l_2$
Cloning Items
Use of Pullbacks

Cloning the subgraph containing nodes $l_0, l_1, l_2$
Graph Rewriting

Give three rules implementing the following evolution
Graph Rewriting

Exercise

Starting from a graph $G$ modeling agents $(A)$, files $(F)$ and an arbitrary access relation $(R)$ including possible prohibited accesses $(R \subseteq A \times F)$, give a rewrite rule which transforms $G$ into a graph that satisfies the following policy.
There are two responsibility levels among agents: H and L. Files are classified according to 3 security levels: 1, 2 and 3. Agents of responsibility level H have the right to access files of security levels 1 and 2. Agents of responsibility level L have the right to access files of security levels 2 and 3.
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DPO Rules
First things first!


\[ L \leftarrow K \rightarrow R \]
DPO Rules
First things first!


\[ L \xleftarrow{m} POC \xrightarrow{l} K \xrightarrow{d} PO \xrightarrow{r} R \xrightarrow{m'} H \xleftarrow{l'} D \xrightarrow{r'} \]

A DPO rewrite step:
POCs (Pushout complement) are not unique when cloning items!
Delete actions are restricted by the gluing conditions.
SQPO Rules

A SQPO rewrite step:

\[ L \leftarrow K \rightarrow R \]

\[ L \leftarrow^l K \rightarrow^r R \]
\[ G \leftarrow^{l'} D \rightarrow^{r'} H \]

Clone action is still quite limited!
Caution: The definition of AGREE transformation requires the existence, in the underlying category, of a partial map classifier [ICGT 2015][TCS 2019,to appear]
AGREE Rules

An example
Clone action is more flexible than SQPO but can still be improved!
A PBPO rule consists of a (classical) first span of the form:

\[ L \leftarrow K \rightarrow R \]

to which it is added a (typing) second span

\[ L' \leftarrow K' \rightarrow R' \]

such that the two following squares commute:

\[
\begin{array}{c}
L \leftarrow K \rightarrow R \\
\downarrow t_L \hspace{1cm} = \hspace{1cm} \downarrow t_K \hspace{1cm} = \hspace{1cm} \downarrow t_R \\
L' \leftarrow K' \rightarrow R'
\end{array}
\]

PBPO

A rewrite step

PBPO rule:

\[
\begin{align*}
L & \leftarrow^l K & R & \rightarrow^r K' & R' \\
\downarrow t_L & = & \downarrow t_K & = & \downarrow t_R \\
L' & \leftarrow^{l'} K' & \rightarrow^{r'} R' 
\end{align*}
\]

PBPO rewrite step: The match is defined as a pair \((m, m')\)!
PBPO Rewrite Step

Example
The PBPO Approach:

Exercise

Give a rewrite rule that makes a copy of the pages of a local web site or a copy of a whole directory

```
cp -r <directory> <new directory>
```

```
cp <a local web site>
```

AGREE needs a new rule for every specific shape of the web site
PBPO uses only one generic rule!
PBPO Rewrite Step

Example of the copy of local Web pages
**Proposition**

Let $\alpha$ be an AGREE rule in a category with a partial map classifier. Then there is a PBPO rule $\rho_\alpha$ such that for each mono $m : L \rightarrow G$ we have $G \Rightarrow^\text{AGREE}_\alpha H$ if and only if $G \Rightarrow^\text{PBPO}_\alpha H$ using match $(m, \overline{m})$ with $\overline{m} : G \rightarrow T(L)$.

Add $R'$ as a Pushout of morphisms $t$ and $r$ to end the construction!
PBPO vs SQPO

Proposition

Let $\alpha$ be a SQPO rule in a category with a partial map classifier. Then there is a PBPO rule $\rho_\alpha$ such that for each mono $m : L \to G$ we have $G \Rightarrow^\alpha_{\text{SQPO}} H$ if and only if $G \Rightarrow^{\rho_\alpha}_{\text{PBPO}} H$ using match $(m, \overline{m})$ with $\overline{m} : G \to T(L)$.

Add $R'$ as a Pushout of morphisms $t$ and $r$ to end the construction!
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Let **Graph** be a category of structures (e.g., graphs)
Let **Att** be a category of attribute structures (e.g., $\Sigma$-algebras)
Let $S : \textbf{Graph} \rightarrow \textbf{Set}$ be a functor
Let $T : \textbf{Att} \rightarrow \textbf{Set}$ be a functor

**Definition**

The category $\textbf{AttG}$ of attributed graphs is defined as the comma category $S \downarrow T$. 
Attributed Graphs

- Let $S : \text{Graph} \to \text{Set}$ be a functor
- Let $T : \text{Att} \to \text{Set}$ be a functor

**Attributed Graph** : $\hat{G} = (G, A, \alpha)$
- $G$ in $\text{Graph}$,
- $A$ in $\text{Att}$ and
- $\alpha : S(G) \to T(A)$ (in $\text{Set}$) is a labelling function

**Morphisms** : $\hat{g} : \hat{G} \to \hat{G}'$, where $\hat{G} = (G, A, \alpha)$ and $\hat{G}' = (G', A', \alpha')$, is a pair $\hat{g} = (g, a)$ with $g : G \to G'$ is a morphism in $\text{Graph}$ and $a : A \to A'$ is a morphism in $\text{Att}$ such that $\alpha' \circ Sg = Ta \circ \alpha$ (in $\text{Set}$).

\[
\begin{array}{cccccc}
\hat{G} & \downarrow \hat{g} & \Rightarrow & G & \downarrow g & \Rightarrow & SG & \xrightarrow{\alpha} & TA & \Rightarrow & A \\
\hat{G}' & \downarrow & \Rightarrow & G' & \downarrow & \Rightarrow & SG' & \xrightarrow{\alpha'} & TA' & \Rightarrow & A'
\end{array}
\]
Partially Attributed Graphs

- **Partially Attributed Graph**: \( \hat{G} = (G, A, \alpha) \)
  - \( G \) in \( \text{Graph} \),
  - \( A \) in \( \text{Att} \) and
  - \( \alpha : S_p(G) \rightarrow T_p(A) \) (in \( \text{Pfn} \)) is a partial labeling function

- **Morphisms**: \( \hat{g} : \hat{G} \rightarrow \hat{G}' \), where \( \hat{G} = (G, A, \alpha) \) and \( \hat{G}' = (G', A', \alpha') \), is a pair \( \hat{g} = (g, a) \) with \( g : G \rightarrow G' \) is a morphism in \( \text{Graph} \) and \( a : A \rightarrow A' \) is a morphism in \( \text{Att} \) such that \( \alpha' \circ S_p g \geq T_p a \circ \alpha \) (in \( \text{Pfn} \)).

\[
\begin{array}{cccccc}
\hat{G} & \downarrow \hat{g} & = & G & \downarrow g & \text{Sp} G & \xrightarrow{\alpha} & T_p A & \downarrow a \\
\hat{G}' & \downarrow g & & G' & \downarrow \text{Sp} g & \geq & T_p A' & \downarrow a \\
\end{array}
\]

Remark: \( \geq \) states that morphisms preserve defined attributes
A morphism of partially attributed structures \( (g, a) \) is called strict when \( \alpha' \circ S_p g = T_p a \circ \alpha \).
with the additional conditions

- $\alpha_L, \alpha'_L, \alpha_R$ and $\alpha'_R$ are total labeling functions
- The morphism $\hat{t}_K$ is strict
- The morphism $\hat{t}_K$ is injective on non-attributed items
Does H always exist?
Is H completely attributed?
PBPO Rewrite Step

Easy examples

```
n: x ← n: x → n: x
      ↓     ↓     ↓
n: 6 ← n: 6 → n: 6
      ↓     ↓     ↓
n: nat ← n: nat → n: nat
```

Node $n$ is preserved together with its attribute
PBPO Rewrite Step

Easy examples

Node $n$ is preserved but re-attributed
Case of a non strict $\hat{t}_K$
Case where $\hat{t}_K$ is not injective on non-attributed items
Case where a non-attributed element in $\hat{K}'$, $n'$, has no antecedent in $\hat{K}$.
Existence and Total Attribution of Transformed Graphs

\[ (L, A, \alpha_L) \xleftarrow{(l, id_A)} (K, A, \alpha_K) \xrightarrow{(r, id_A)} (R, A, \alpha_R) \]

\[ (G, A_0, \alpha_G) \xleftarrow{(g, id_{A_0})} (D, A_0, \alpha_D) \xrightarrow{h} (H, A_0, \alpha_H) \]

\[ (L', A', \alpha_{L'}) \xleftarrow{(l', id_{A'})} (K', A', \alpha_{K'}) \xrightarrow{(r', id_{A'})} (R', A', \alpha_{R'}) \]

Proposition

If the following conditions hold

- \( \alpha_L, \alpha'_L, \alpha_R \) and \( \alpha'_R \) are total labeling functions
- The morphism \( \hat{t}_K \) is strict and injective on non-attributed items
- \( G \) is completely attributed
- \( \forall n \in G, \text{ if } \exists n_{K'} \in K' \text{ such that } n_{K'} \text{ is not attributed and } m'(n) = l'(n_{K'}), \text{ then } \exists n_k \in K \text{ with } n = m(l(n_K)) \text{ and } n_{K'} = t_K(n_K). \)

Then the graph \( H \) exists and is completely attributed
Termgraph Rewriting

Motivation

- Handling **Data-structure rewriting**
  including **cyclic data-structures with pointers**
  such as circular lists, doubly-linked lists, etc.
- Data-structures are more complex than terms (**Cycles, Sharing**)
- Difficult to encode efficiently using terms
- Usually described by pointers (**pointer rewriting**)
- Formally described as **termgraphs**
  Informally: termgraph = term with cycles and sharing
Termgraph Rewriting

Motivation

\[ 0 + x \rightarrow x \]
\[ s(x) + y \rightarrow s(x + y) \]
\[ \text{double}(x) \rightarrow x + x \]

Term rewrite systems constitute a very well established domain with several results: Confluence, Termination, Strategies, Proof methods (equational reasoning, induction) etc. However, subterm sharing, as in termgraph, does not preserve classical properties of term rewriting such as, e.g., the confluence property.
Sharing Subterms (information) and Term Rewriting

Consider the following rules:

\[ f(a, b) \rightarrow c \]
\[ a \rightarrow b \]

Sharing does not preserve properties of tree (term) rewriting!

Term rewrite derivation: \( f(a, a) \rightarrow f(a, b) \rightarrow c \)

Termgraph rewrite derivation:

\[\begin{array}{c}
\text{Termgraph rewrite derivation:} \\
\downarrow \quad \downarrow \\
\quad f \quad \rightarrow \\
\quad a \\
\quad \downarrow \\
\quad \quad b
\end{array}\]

[Plump 99] survey on rewriting with “dags”.

Termgraphs
[Barendregt et al. 87]
[Plump 99, survey on *acyclic* term-graphs]

Let $\Omega$ be a set of operation symbols. A *term-graph* $t$ over $\Omega$ is defined by:

- a set of nodes $N_t$,
- a subset of labeled nodes $N^\Omega_t \subseteq N_t$,
- a labeling function $\mathcal{L}_t : N^\Omega_t \rightarrow \Omega$,
- a successor function $S_t : N^\Omega_t \rightarrow N^*_t$,

![Diagram of term-graph](image-url)
Termgraphs

[Barendregt et al. 87]
[Plump 99, survey on acyclic term-graphs]

Let $\Omega$ be a set of operation symbols and $\mathcal{F}$ a set of feature symbols. A term-graph $t$ over $\Omega$ and $\mathcal{F}$ is defined by:

- a set of nodes $\mathcal{N}_t$,
- a set of edges $E_t$
- a subset of labeled nodes $\mathcal{N}_t^\Omega \subseteq \mathcal{N}_t$,
- a node labeling function $\mathcal{L}_t^n : \mathcal{N}_t^\Omega \rightarrow \Omega$,
- an edge labeling function $\mathcal{L}_t^e : E_t \rightarrow \mathcal{F}$
- a source function $S_t : E_t \rightarrow \mathcal{N}_t$,
- a target function $T_t : E_t \rightarrow \mathcal{N}_t$, 
Algorithmic approach

[Barendregt et al. 87]

Shape of a rule:

\[ L \rightarrow R \]

where \( L \) and \( R \) are rooted term-graphs.

A rule can be defined as one graph together with two roots

\[(L + R, r_1, r_2)\]

where \( r_1 \) and \( r_2 \) are the roots of \( L \) and \( R \) respectively.

Let \( \rho \) be the rule \((L + R, r_1, r_2)\)

We say that \( G \) rewrites to \( H \) using the rule \( \rho \) if

- L matches a subgraph of \( G \) (\( h : L \rightarrow G \mid_n \))
- (build phase) Construct graph \( G_1 = G + h(R) \)
- (redirection phase) \( G_2 = [h(r_1) \gg h(r_2)]G_1 \)
- (garbage collection phase) \( H = G_2 \mid_{\text{root}} \)

A cumbersome definition, hard to deal with in practice!
Rewrite Rules with actions

Shape of a rewrite rule:

\[ [L \mid C] \rightarrow R \]

- \( L \) is a term-graph pattern
- \( C \) is a node constraint, \( \bigwedge_{i=1}^{n} (\alpha_i \not\approx \beta_i) \).
- \( R \) is a sequence of actions \( a_1; a_2; \ldots; a_n \).
We consider three kinds of actions:

- **Node definition** $\alpha : f(\alpha_1, \ldots, \alpha_n)$
- **Edge redirection** $\alpha \triangleright_i \beta$
- **Global redirection** $\alpha \triangleright \beta$
Application of actions

$a[t]$ denotes the application of action(s) $a$ to the termgraph $t$

- Let $t = n: f(p, q : a)$

- Let $t_1 = p : h(p)[t] = n : f(p : h(p), q : a)$
Application of actions

\( a[t] \) denotes the application of action(s) \( a \) to the term graph \( t \)

- Let \( t_1 = p : h(p)[t] = n : f(p : h(p), q : a) \)

- Let \( t_2 = n \gg_2 p[t_1] = n : f(p : h(p), p); q : a \)
Application of actions

$a[t]$ denotes the application of action(s) $a$ on the term-graph $t$

- Let $t_2 = n \gg_2 p[t_1] = n: f(p:h(p), p); q : a$

- Let $t_3 = p \gg q[t_2] = n: f(q, q); p: h(q)$
Rewrite Step

Let $t$ be a termgraph

Let $\rho$ be a rewrite rule $[L \mid C] \rightarrow R$

$t$ rewrites to $s$ at node $\alpha$, $t \rightarrow_\alpha s$ iff:

- $\exists m : L \rightarrow t$ a homomorphism
- $m(root_L) = \alpha$
- $\alpha$ is reachable from $root_t$
- $m(C)$ holds
- $s = m(R)[t]$
Termgraph Rewrite Systems (tGRS)

Example

Length of a circular list:

\[ r : \text{length}(p) \rightarrow r : \text{length}'(p, p) \]
\[ r : \text{length}'(p_1 : \text{cons}(n, p_2), p_2) \rightarrow r : s(0) \]
\[ [r : \text{length}'(p_1 : \text{cons}(n, p_2), p_3) \mid p_2 \not\approx p_3] \rightarrow r : s(q); q : \text{length}'(p_2, p_3) \]

Remark: term rewrite systems are tGRS’s.
Termgraph Rewrite Systems
Example

In-situ list reversal:

\[ o : \text{reverse}(p) \rightarrow o : \text{rev}(p, \text{nil}) \]

\[ o : \text{rev}(p_1 : \text{cons}(n, \text{nil}), p_2) \rightarrow p_1 \gg 2 p_2; o \gg p_1 \]

\[ o : \text{rev}(p_1 : \text{cons}(n, p_2 : \text{cons}(m, p_3), p_4) \rightarrow p_1 \gg 2 p_4; o \gg 1 p_2; o \gg 2 p_1 \]

Visual Programming would help!
DPO approach of rewrite rules with actions

A categorical approach can be found in [TERMGRAPH 06, ENTCS07, RTA07]

![Diagram](image-url)

**Figure:** Double pushout: a rewrite step ($G \rightarrow H$)

Redirections of edges (pointers) are handled by $K = disconnection(L, E, N)$ and the morphisms $l$ and $r$.

**Remark:** Morphisms $l$ and $r$ are not injective! $D$ is not unique!
The following term-graph:

\[ f(x) \to x \]
\[ g(x) \to x \]

rewrites to:

\[ f(x) \to x \]
\[ g(x) \to x \]
Confluence

\[ \alpha : f(\beta : c) \rightarrow \beta : a; \alpha \gg \beta \]

\[ \alpha : g(\beta : c) \rightarrow \beta : b; \alpha \gg \beta \]

\[ \begin{array}{c}
p : f \\
\downarrow \\
q : c \\
\end{array} \quad \begin{array}{c}
q : g \\
\downarrow \\
p : f \\
\end{array} \]

The label of node \( q \) may end as \( q : a \) or \( q : b \)
How to evaluate the following termgraph?

- \( \text{addlast}(\text{length}(n : [1, 2]), n) \)
- Two normal forms
  - \([1, 2, 2]\) (evaluate \(\text{addlast}\) after \(\text{length}\))
  - \([1, 2, 3]\) (evaluate \(\text{length}\) after \(\text{addlast}\))
Termgraphs with Priority

[PPDP06][RTA07][RTA08]

- Endow Termgraphs with priorities \((G, <_G)\) to express which node should be evaluated first
  - \(m_1 : addlast(m_2 : length(n : [1, 2]), n); m_1 < m_2\)
- Priorities should not be a total order (stay declarative)
- Which nodes should be ordered?
- Solution: Order only nodes producing a “side-effect”
A **strategy** $\phi$ is a partial function which takes a rooted termgraph $t$ and returns a node (position) $n$ and a rule $R$,

$$\phi(t) = (n, R)$$

such that the termgraph $t$ can be reduced at node $n$ using the rule $R$,

$$t \rightarrow_n t'$$
Let $\phi$ be a rewrite strategy. Let $\phi(t) = (p, R)$. The node $p$ is needed iff for all derivations

$$t \rightarrow^{\beta_1} t_1 \rightarrow^{\beta_2} \cdots t_{n-1} \rightarrow^{\beta_n} t_n$$

such that $t_n$ is a value, there exists $i \in [1..n]$ s.t. $\beta_i = p$
Inductively sequential Term Rewrite Systems

- Constitute a subclass of TRSs for which efficient rewrite strategies are available [Antoy 92]
- Are as expressive as Strongly Sequential TRSs
- Are defined by means of data-structures called Definitional trees
Definitional Trees - case of terms -

Let $\mathcal{R}$ be the following TRS

\[
\begin{align*}
    f(k,\text{nil}) & \rightarrow R1 \\
    f(0,\text{cons}(x,l)) & \rightarrow R2 \\
    f(\text{succ}(n),\text{cons}(x,l)) & \rightarrow R3
\end{align*}
\]

A definitional tree of operator $f$ is a hierarchical structure whose leaves are the rules defining $f$. 

\[
\begin{align*}
    f(k, l) \\
    & \quad f(k, \text{nil}) \rightarrow R1 \\
    & \quad f(k, \text{cons}(x,u)) \\
    & \qquad f(0, \text{cons}(x,u)) \rightarrow R2 \\
    & \qquad f(\text{succ}(y), \text{cons}(x,u)) \rightarrow R3
\end{align*}
\]
Definitional trees
-case of termgraphs-

$r : \text{length'}(p_1 : \text{nil}, p_2 : \bullet) \rightarrow \text{rhs}_1$

$r : \text{length'}(p_1 : \text{cons}(n : \bullet, p_2 : \bullet), p_2) \rightarrow \text{rhs}_2$

$[r : \text{length'}(p_1 : \text{cons}(n : \bullet, p_2 : \bullet), p_3 : \bullet) \mid p_2 \neq p_3] \rightarrow \text{rhs}_3$

A definitional tree $T$ of the operation $\text{length'}$ is given bellow:

$r : \text{length'}(p_1 : \bullet, p_2 : \bullet)$

\begin{align*}
  r & : \text{length'}(p_1 :\text{nil}, p_2 : \bullet) \rightarrow \text{rhs}_1 \\
  r & : \text{length'}(p_1 : \text{cons}(n : \bullet, p_3 : \bullet), p_2 : \bullet) \\
  & \quad r : \text{length'}(p_1 : \text{cons}(n : \bullet, p_2 : \bullet), p_2) \rightarrow \text{rhs}_2 \\
  & \quad [r : \text{length'}(p_1 : \text{cons}(n : \bullet, p_2 : \bullet), p_3 : \bullet) \mid p_2 \neq p_3] \rightarrow \text{rhs}_3
\end{align*}
A Rewrite strategy $\phi$

Consider the following definitional tree $T$ of the operation $g$:

\[ r : g(p_1 : \bullet, p_2 : \bullet) \]
\[ r : g(p_1 : \text{nil}, p_2 : \bullet) \rightarrow \text{rhs}_1 \]
\[ r : g(p_1 : \text{cons}(n : \bullet, p_3 : \bullet), p_2 : \bullet) \]
\[ r : g(p_1 : \text{cons}(n : \bullet, p_2 : \bullet), p_2) \rightarrow \text{rhs}_2 \]
\[ [r : g(p_1 : \text{cons}(n : \bullet, p_2 : \bullet), p_3 : \bullet) \mid p_2 \neq p_3] \rightarrow \text{rhs}_3 \]

$\phi(1 : g(2 : g(3 : g(\text{nil}, p), q), 4 : g(\text{nil}, o))))$

= $\phi(2 : g(3 : g(\text{nil}, p), q))$

= $\phi(3 : g(\text{nil}, p))$

= $(3, \text{Rule1})$
Naive extension of TRS’s

Contrary to term rewriting, Definitional trees are not enough to ensure the neededness of positions computed by the strategy $\phi$, in the context of term-graph rewriting.

**Proposition:** Let $SP = \langle \Omega, R \rangle$ be tGRS such that $\Omega$ is constructor-based and the rules of every defined operation are stored in a definitional tree. Let $t$ be a rooted termgraph. Then,

1. if $\phi(t) = (p, R)$, the node $p$ is not needed in general.
2. if $\phi(t)$ is not defined, $g$ can still have a constructor normal form.
Counter-examples

\[ r : f(p : 0) \rightarrow r \gg p \quad r : h(p : 0, q : succ(n : \bullet)) \rightarrow q \gg p \]
\[ r : f(p : succ(p' : \bullet)) \rightarrow r \gg p \]

Let \( t = n : succ \)

\[ \phi(t) = (p, r : f(p : succ(p' : \bullet)) \rightarrow r \gg p) \]

However, the node \( p \) is not needed in \( t \).
Counter-examples

\[ r : g(p : 0) \rightarrow r \gg p \quad \text{and} \quad r : h(p : 0, q : \text{succ}(n : \bullet)) \rightarrow q \gg p \]

Let \( t = n : \text{succ} \)

\[ \Downarrow \]

\[ r : \text{succ} \]

\[ \Downarrow \]

\[ p : g \]

\[ \Downarrow \]

\[ q : \text{succ} \]

\[ \Downarrow \]

\[ s : h \]

\[ \Downarrow \]

\[ u : 0 \]

\( \phi(t) \) is not defined!

However, the termgraph \( t \) rewrites to \( n : \text{succ}(u : 0) \).
Let $SP = \langle \Omega, \mathcal{R} \rangle$ be a tGRS. $SP$ is called inductively sequential iff

- The rules of every defined operation can be stored in a definitional tree and
- for all rules $[L \mid C] \rightarrow r$ in $\mathcal{R}$, for all global (respectively, local) redirections of the form $p \gg q$ (respectively, $p \gg_i q$ for some $i$), occurring in the right-hand side $r$, $p = \text{Root}_L$.  

(Shaded text indicates the focus of the explanation.)
Main Properties of Strategy $\Phi$

In presence of Inductively Sequential Termgraph Rewrite Systems

- The positions computed by $\Phi$ are needed
- $\Phi$ is c-normalizing
- $\Phi$ is c-hyper-normalizing
- Derivations computed by $\Phi$ have minimal length
Confluence

Inductively sequential tGRS are not confluent!

\[ f(p : \bullet, p) \rightarrow 0 \]
\[ [f(p : \bullet, q : \bullet) \mid p \neq q] \rightarrow 1 \]
\[ r : g(q : \bullet) \rightarrow r \gg q \]

Let \( t = \)

\[ \begin{array}{c}
  n : f \\
  p : g \quad q : 0
\end{array} \]

There are two different derivations starting from \( t : \)

\[ t \rightarrow_n 1 \]
\[ t \rightarrow_p f(q : 0, q) \rightarrow_n 0 \]
Admissible termgraphs

[JICSLP98]
Ω is constructor-based, i.e. \( \Omega = D \cup C \) and \( D \cap C = \emptyset \)

\( D \) is a set of defined operations
\( C \) is a set of constructors

A termgraph is **admissible** if none of its cycles includes a defined operation.

\( n : \text{succ}(n) \) is an admissible termgraph
\( n : + (n, n) \) and \( n : \text{tail}(n) \) are not admissible
Admissible termgraphs

The set of admissible termgraphs is not closed under rewriting

\[ n : f(m) \rightarrow q : g(n); n \gg m \]

Let \( \Omega = D \cup C \) with \( C = \{0, \text{succ}\} \) and \( D = \{f, g\} \)

\[ n_1 : f(m_1 : 0) \rightarrow q_1 : g(q_1) \]
Admissible Inductively sequential Termgraph Rewrite Systems

Let $SP = \langle \Omega, \mathcal{R} \rangle$ be an inductively sequential tGRS. $SP$ is called admissible iff for all rules $[\pi \mid C] \rightarrow r$ in $\mathcal{R}$ the following conditions are satisfied:

- for all global (respectively, local) redirections of the form $p \gg q$ (respectively, $p \gg_i q$ for some $i$), occurring in the right-hand side $r$, we have $p = \text{Root}_\pi$ and $q \neq \text{Root}_\pi$.
- for all actions of the form $\alpha : f(\beta_1, \ldots, \beta_n)$, for all $i \in 1..n$, $\beta_i \neq \text{Root}_\pi$
- the set of actions of the form $\alpha : f(\beta_1, \ldots, \beta_n)$, appearing in $r$, do not construct a cycle including a defined operation.
- Constraint $C$ includes disequations of the form $p \neq q$ where $p$ and $q$ are labeled by constructor symbols.
Admissible Inductively sequential Termgraph Rewrite Systems

[ICGT08][JICSLP98]
In presence of Admissible Inductively sequential Termgraph Rewrite Systems

- The set of admissible termgraphs is closed under the rewrite relation defined by admissible rules.
- $\Phi$ computes needed positions
- Admissible termgraphs admit unique normal forms
Narrowing
Lifting optimal rewrite strategies to narrowing in the case of Admissible termgraph rewrite systems

Let $\mathcal{R}$ be the following TRS
\[
\begin{align*}
\leq (0, y) & \rightarrow \text{true} \\
\leq (s(x), 0) & \rightarrow \text{false} \\
\leq (\text{succ}(x), \text{succ}(y)) & \rightarrow \leq (x, y)
\end{align*}
\]

A definitional tree of operator $\leq$ is as follows:

\[
\begin{align*}
\leq (i, j) \\
\leq (0, j) & \rightarrow \text{true} \\
\leq (s(i_1), j) \\
\leq (s(i_1), 0) & \rightarrow \text{false} \\
\leq (s(i_1), s(j_1)) & \rightarrow \leq (i_1, j_1)
\end{align*}
\]
Definitional Trees -case of terms-

\[ \leq (i, j) \]
\[ \leq (0, j) \rightarrow true \]
\[ \leq (s(i_1), j) \]
\[ \leq (s(i_1), 0) \rightarrow false \]
\[ \leq (s(i_1), s(j_1)) \rightarrow \leq (i_1, j_1) \]

How to narrow the expression \( \leq (i, j + k) \)?
\[ \leq (i, j + k) \rightsquigarrow_{j \mapsto 0} \leq (i, k) \rightsquigarrow_{i \mapsto 0} true \]

Remark: The assignment \( j \mapsto 0 \) is useless!

The use of definitional trees prevents non necessary assignments and develops \( \leq (i, j + k) \rightsquigarrow_{i \mapsto 0} true \)

Key idea: Get rid of most general unifiers. Use of definitional trees to make a traversal of term (graphs) and compute only necessary
Some Results

Needed Term narrowing [POPL94][JACM2000]
Needed Graph Narrowing [JICSLP98]
Needed Collapsing Narrowing [Gratra 2000]
Narrowing-based algorithm for data-structure rewriting [ICGT06]

- **Goal**
  \[ o : equal(p : length(q), s(s(0)))) = true \]

- **Solution** : a circular list of length two
  \[ [q : cons(n_1, r : cons(n_2, q)) | q \not\approx r] \]
Outline

1. Introduction
2. Preliminary Definitions
3. Graph Rewriting: Elementary Actions
4. Some Algebraic Approaches to Graph Rewriting
5. Attributed Graph Transformation and PBPO rules
6. Termgraph Rewriting: An Algorithmic Approach
7. Verification of Graph Transformation
Partial Correctness à la Hoare of Graph Rewrite Systems

To be proven: \{Pre(input)\} \quad Program \quad \{Post(output)\}

- **Program** is a graph or model transformation system
- **input** and **output** are graphs or models
- **Pre** and **Post** are formulas, of a given logic $\mathcal{L}$, over the inputs and the outputs
Logically Decorated Graphs

Let $\mathcal{L}$ be a set of formulas, a logically decorated graph $G$ is a tuple $(N, E, \lambda_N, \lambda_E, s, t)$ where:

- $N$ is a set of nodes,
- $E$ is a set of edges,
- $\lambda_N : N \rightarrow 2^\mathcal{L}$ is a node labeling function,
- $\lambda_E : E \rightarrow \mathcal{L}$ is an edge labeling function
- source and target functions: $s : E \rightarrow N$ and $t : E \rightarrow N$

In this talk, the set $\mathcal{L}$ consists of description logic (DL) formulas.
Why considering Description Logics (DLs)?

- DLs constitute a formal basis of knowledge representation languages.
- DLs provide logical basis for ontologies. (E.g., the web ontology language OWL is based on DLs)
- Reasoning problems for DLs are decidable (in general)
DL Syntax

a DL syntax allows one to define:

- **Concept** names, which are equivalent to classical first-order logic unary predicates,

- **Role** names, which are equivalent to binary predicates and

- **Individuals**, which are equivalent to classical constants.

There are various DLs in the literature, they mainly differ by the logical operators they offer to construct concept and role expressions or axioms.
Let $C_0$ (resp. $R_0$ and $O$) be a set of atomic concepts (resp. atomic roles and nominals).
Let $c_0 \in C_0$, $r_0 \in R_0$, $o \in O$, and $n$ an integer.

The set of concepts $C$ and roles $R$ are defined by:

\[
C := \top | c_0 | \exists R.C | \neg C | C \lor C \\
| o \text{ (nominals, } O) \\
| \exists R.\text{Self} \text{ (self loops, } \text{Self}) \\
| (< n R C) \text{ (counting quantifiers, } Q)
\]

\[
R := r_0 \\
| U \text{ (universal role, } U) \\
| R^- \text{ (inverse role, } I)
\]

Examples of DL logics: $\text{ALC}$, $\text{ALCUO}$, $\text{ALCUI}$, …
Examples of some requirements about the organization of a hospital:

- **All patients of a pediatrician are children:**
  
  - **First-order formula:**
    \[ \forall x, y. \text{Pediatrician}(x) \land \text{Has\_patient}(x, y) \Rightarrow \text{Child}(y) \]
  
  - **DL formula \( ALCU \):**
    \[ \forall U. \text{Pediatrician} \Rightarrow \forall \text{Has\_patient}. \text{Child} \]
Examples of properties

Examples of some requirements about the organization of a hospital:

- **All patients of a pediatrician are children:**
  
  First-order formula:
  \[ \forall x, y. \text{Pediatrician}(x) \land \text{Has\_patient}(x, y) \Rightarrow \text{Child}(y) \]
  
  DL formula \((\mathcal{ALCU})\):
  \[ \forall U. \text{Pediatrician} \Rightarrow \forall \text{Has\_patient}. \text{Child} \]

- **Dr. Smith is a pediatrician:**
  
  First-order formula:
  \[ \exists x. \text{Dr. Smith} \equiv x \land \text{Pediatrician}(x) \]
  
  DL formula \((\mathcal{ALCUO})\):
  \[ \exists U. \text{Dr. Smith} \land \text{Pediatrician} \]
Examples of some requirements about the organization of a hospital:

- **All patients of a pediatrician are children:**
  - First-order formula:
    $\forall x, y. \text{Pediatrician}(x) \land \text{Has\_patient}(x, y) \Rightarrow \text{Child}(y)$
  - DL formula ($\mathcal{ALCU}$): $\forall U. \text{Pediatrician} \Rightarrow \forall \text{Has\_patient}. \text{Child}$

- **Dr. Smith is a pediatrician:**
  - First-order formula: $\exists x. \text{Dr}.Smith = x \land \text{Pediatrician}(x)$
  - DL formula ($\mathcal{ALCUO}$): $\exists U. \text{Dr}.Smith \land \text{Pediatrician}$

- **All patients are a doctor’s patients:**
  - First-order formula:
    $\forall x, y. \text{Patient}(x) \Rightarrow \text{Has\_patient}(y, x) \land \text{Doctor}(y)$
  - DL formula ($\mathcal{ALCUI}$): $\forall U. \text{Patient} \Rightarrow \exists \text{Has\_patient}^\complement. \text{Doctor}$
Examples of properties (Continued)

Examples of some requirements about the organization of a hospital:

1. An operation can only be associated with one operating room:
   First-order formula:
   \[ \forall x, y, z. \text{Operation}(x) \land \text{Scheduled_in}(x, y) \land \text{Scheduled_in}(x, z) \land \text{Operation_room}(y) \land \text{Operation_room}(z) \Rightarrow y = z \]
   DL formula \((ALCUQ)\):
   \[ \forall U. \text{Operation} \Rightarrow (\leq 2 \text{Scheduled_in.Operation_room}) \]
Examples of properties (Continued)

Examples of some requirements about the organization of a hospital:

1. An operation can only be associated with one operating room:
   First-order formula:
   \[ \forall x, y, z. \text{Operation}(x) \land \text{Scheduled\_in}(x, y) \land \text{Scheduled\_in}(x, z) \land \text{Operation\_room}(y) \land \text{Operation\_room}(z) \Rightarrow y = z \]
   DL formula (\(\text{ALCUQ}\)):
   \[ \forall U. \text{Operation} \Rightarrow (\langle 2 \text{Scheduled\_in.\text{Operation\_room}} \rangle) \]

2. A doctor cannot be his/her own patient:
   First-order formula: \[ \forall x. \text{Doctor}(x) \Rightarrow \neg \text{Has\_patient}(x, x) \]
   DL formula (\(\text{ALCUQ}\)): \[ \forall U. \text{Doctor} \Rightarrow \neg \exists \text{Has\_patient.\text{SELF}} \]
Examples of properties (Continued)

Examples of some requirements about the organization of a hospital:

1. **An operation can only be associated with one operating room:**
   First-order formula:
   \[ \forall x, y, z. \text{Operation}(x) \land \text{Scheduled\_in}(x, y) \land \text{Scheduled\_in}(x, z) \land \text{Operation\_room}(y) \land \text{Operation\_room}(z) \Rightarrow y = z \]
   DL formula \((\text{ALCUQ})\):
   \[ \forall U. \text{Operation} \Rightarrow (\langle < 2 \text{Scheduled\_in.\_Operation\_room} \rangle) \]

2. **A doctor can not be his/her own patient:**
   First-order formula: \( \forall x. \text{Doctor}(x) \Rightarrow \neg \text{Has\_patient}(x, x) \)
   DL formula \((\text{ALCUQ})\): \( \forall U. \text{Doctor} \Rightarrow \neg \exists \text{Has\_patient.\_SELF} \)

3. **Only ”private” nodes can have access to ”private” nodes. Public ”nodes” cannot have access to ”private” nodes:**
   First-order formula:
   \[ \forall x. y. (\text{Has\_access}(x, y) \land \text{Private}(y)) \Rightarrow \text{Private}(x) \]
   DL formula \((\text{ALCUI})\): \( \forall U. \text{Private} \Rightarrow \forall \text{Has\_access}^-.\text{Private} \)

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The considered Graph Rewriting rules are of the form $L \rightarrow R$ where:

- $L$ is a graph
- $R$ is a sequence of elementary actions
Some Elementary Actions

Let $C_0$ (resp. $R_0$) be a set of node (resp. edge) labels. An elementary action, say $a$, may be of the following forms:

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- an edge addition $\text{add}_E(e, i, j, r)$ (resp. edge deletion $\text{del}_E(e, i, j, r)$) where $e$ is an edge, $i$ and $j$ are nodes and $r$ is an edge label in $R_0$. 
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- a *merge action* $\text{mrg}(i, j)$ where $i$ and $j$ are nodes.
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- a merge action $\text{mrg}(i, j)$ where $i$ and $j$ are nodes.
- a clone action $\text{cl}(i, j, L_{\text{in}}, L_{\text{out}}, L_{\text{lin}}, L_{\text{lout}}, L_{\text{lloop}})$ where $i$ and $j$ are nodes and $L_{\text{in}}, L_{\text{out}}, L_{\text{lin}}, L_{\text{lout}}$ and $L_{\text{lloop}}$ are subsets of $R_0$. It clones a node $i$ by creating a new node $j$ and connects $j$ to the rest of a host graph according to different information given in the parameters $L_{\text{in}}, L_{\text{out}}, L_{\text{lin}}, L_{\text{lout}}, L_{\text{lloop}}$. 
Graph Rewrite Systems: Example

\[ \rho_0: (I : LLIN \land \exists ins_in. \top \times has_ins \rightarrow i : DDT) \rightarrow del_N(I) \]

\[ \rho_1: (I : LLIN \times has_ins \xrightarrow{e : ins_in} i : Insecticide) \rightarrow (I' : LLIN \times has_ins \xrightarrow{e : ins_in} i' : Insecticide)
\quad \text{has_moa} \xrightarrow{m : ModeOfAction} \quad \text{has_moa} \xrightarrow{m' : ModeOfAction \land \neg m}
\]

\[ cl(I', I'', \hat{L}); del_E(e, I, h, ins_in); add_E(e', I'', h, ins_in) \]

\[ \rho_2: (I : LLIN \land \forall ins_in. \bot \times has_ins \rightarrow i : Insecticide) \rightarrow (I' : LLIN \land \forall ins_in. \bot \land \neg I \times has_ins \rightarrow i' : Insecticide)
\]

\[ mrg(I, I') \]
To be able to apply rules, we need to define when they can be applied.
**Match**

**Definition: Match**

A *match* $h$ between a lhs $L$ and a graph $G$ is a pair of functions $h = (h^N, h^E)$, with $h^N : N^L \rightarrow N^G$ and $h^E : E^L \rightarrow E^G$ such that:

1. $\forall e \in E^L, s^G(h^E(e)) = h^N(s^L(e))$
2. $\forall e \in E^L, t^G(h^E(e)) = h^N(t^L(e))$
3. $\forall n \in N^L, \forall c \in \lambda^L_N(n), h^N(n) \models c$
4. $\forall e \in E^L, \lambda^G_E(h^E(e)) = \lambda^L_E(e)$

**Remark:** The third condition says that for every node, $n$, of the lhs, the node to which it is associated, $h(n)$, in $G$ has to satisfy every concept in $\lambda^L_N(n)$. This condition clearly expresses additional negative and positive conditions which are added to the “structural” pattern matching.
Rewrite Step and Rewrie Derivation

Rewrite step
Let \( \rho = L \rightarrow R \) be a rule and \( G \) and \( G' \) be two graphs. \( G \) rewrites into \( G' \) using rule \( \rho \), noted \( G \rightarrow_\rho G' \) iff:

- There exists a match \( h \) from the left-hand side \( L \) to \( G \), and
- \( G \sim_{h(R)} G' \). I.e., \( G' \) is the result of performing \( h(R) \) on \( G \)

Rewrite derivation
Let \( \mathcal{R} \) be graph transformation system and \( G \) and \( G' \) be two graphs.
A rewrite derivation from \( G \) to \( G' \), noted \( G \rightarrow_\mathcal{R} G' \), is a sequence \( G \rightarrow_\rho_0 G_1 \rightarrow_\rho_1 \ldots \rightarrow_\rho_n G' \) such that \( \forall i. \rho_i \in \mathcal{R} \).
A strategy is a word of the following language defined by \( s ::= \)

- \( \rho \) (application of a rule)
- \( s; s \) (sequential composition of strategies)
- \( s \oplus s \) (non-deterministic choice between two strategies)
- \( s^* \) (iteration as long as possible of a strategy)
- \( \ldots \)
A **strategy** is a word of the following language defined by $s ::=$

- $\rho$ (application of a rule)
- $s; s$ (sequential composition of strategies)
- $s \oplus s$ (non-deterministic choice between two strategies)
- $s^*$ (iteration as long as possible of a strategy)
- $\ldots$

**Example:** Strategy $\textit{strat} = s_0; s_1^*; s_2$ performs once the sub-strategy $s_0$, iterates as much as possible sub-strategy $s_1$, before performing once sub-strategy $s_2$. 
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A derivation $G \xrightarrow{\rho_0} G_1 \xrightarrow{\rho_1} \ldots \xrightarrow{\rho_n} G'$ is controlled by a strategy $strat$ iff the word $\rho_0 \rho_1 \ldots \rho_n$ belongs to the language defined by strategy $strat$. 
A specification $\text{spec}$ is a triple $(\text{Pre}, \text{strat}, \text{Post})$ where:

- $\text{Pre}$ is a DL formula called the **precondition**
- $\text{strat}$ is a strategy with respect to a graph transformation system $\mathcal{R}$
- $\text{Post}$ is a DL formula called the **postcondition**.

A specification $\text{spec} = (\text{Pre}, \text{strat}, \text{Post})$ is said to be **correct** iff:

- for all graphs $G$,
- for all graphs $G'$ such that $G \xrightarrow{\text{strat}} G'$
- if $G \models \text{Pre}$ then $G' \models \text{Post}$
Floyd-Hoare Logics

- Let $\mathcal{R}$ be a graph transformation system
- Let $strat$ be a strategy and $\rho_0 \ldots \rho_{n-1} \rho_n$ an element of $strat$
- Let $Pre$ and $Post$ be two DL formulas
- **Aim:** Prove that specification $spec = (Pre, strat, Post)$ is correct

```plaintext
Pre
$\rho_0$;
...
$\rho_{n-1}$;
$\rho_n$;
Post
```
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$\text{Pre}$
$a_0$;
...

$a_{m-1}$;

$a_m$;

$\text{Post}$
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- **Aim:** Prove that specification $\textit{spec} = (\textit{Pre}, \textit{strat}, \textit{Post})$ is correct

```
Pre
  $a_0$;
...
  $a_{m-1}$;
Post[$a_m$]
a_m;
Post
```
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- Let $Pre$ and $Post$ be two DL formulas
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\[
\begin{align*}
Pre & a_0; \\
& \ldots \\
Post[a_m][a_{m-1}] & a_{m-1}; \\
Post[a_m] & a_m; \\
Post & \end{align*}
\]
Floyd-Hoare Logics

- Let $\mathcal{R}$ be a graph transformation system
- Let $strat$ be a strategy and $\rho_0 \ldots \rho_{n-1}\rho_n$ an element of $strat$
- Let $Pre$ and $Post$ be two DL formulas
- **Aim**: Prove that specification $spec = (Pre, strat, Post)$ is correct

$$Pre \Rightarrow Post[a_m][a_{m-1}]\ldots[a_0]$$
$$a_0;$$
$$...$$
$$Post[a_m][a_{m-1}]$$
$$a_{m-1};$$
$$Post[a_m]$$
$$a_m;$$
$$Post$$
Definition: Substitution
A \textit{substitution}, written \([a]\), is associated to each elementary action \(a\), such that for all graphs \(G\) and DL formulas \(\phi\),
\((G \models \phi[a]) \iff (G' \models \phi)\) where \(G'\) is obtained from \(G\) after application of action \(a\), i.e., \(G \leadsto_a G'\).

\[
\begin{align*}
G & \leadsto_a G' \\
\phi[a] & \equiv \phi
\end{align*}
\]
Generating Weakest Preconditions

We define $wp(a, Q)$ the weakest precondition for an elementary action $a$ and a formula $Q$.

$$\downarrow wp(a, Q) = Q[a]$$
Generating Weakest Preconditions

We define \( wp(a, Q) \) the weakest precondition for an elementary action \( a \) and a formula \( Q \).

\[ wp(a, Q) = Q[a] \]  

How to handle substitutions?
Floyd-Hoare Logics: a classical example

The assignment instruction (action)

Weakest precondition: $wp(x := X + 1, Post) \equiv x > 5[x := X + 1]$

Action: $x := x + 1$;

Post: $Post \equiv x > 5$
Floyd-Hoare Logics: a classical example

The assignment instruction (action)

\[ wp(x := X + 1, Post) \equiv x > 5[x := X + 1] \equiv x > 4 \]

Action: \( x := x + 1; \)

Post: \( Post \equiv x > 5 \)
Floyd-Hoare Logics: a basic case

\[ wp(Add_E(e, a, b, R), Post) \equiv \exists U. (a \land (\succ 5R. \top))[Add_E(e, a, b, R)] \]

Action: \( Add_E(e, a, b, R) \);

Post: \( \exists U. (a \land (\succ 5R. \top)) \)
Floyd-Hoare Logics: a basic case

\[ wp(Add_E(e, a, b, R), \text{Post}) \equiv \exists U. (a \land (> 5R. \top)) [Add_E(e, a, b, R)] \equiv (\exists U. (a \land \exists R. b) \Rightarrow \exists U. (a \land (> 5R. \top))) \land \]
\[ (\exists U. (a \land \forall R. \neg b) \Rightarrow \exists U. (a \land (> 4R. \top))) \]

Action: \( Add_E(e, a, b, R) \);

Post: \( \exists U. (a \land (> 5R. \top)) \)
Closure Under Substitutions

A logic \( \mathcal{L} \) is said to be **closed under substitution** iff for every formula \( \phi \in \mathcal{L} \), every substitution \([a] \), \( \phi[a] \in \mathcal{L} \).
Theorem: The description logics $\text{ALCUO}, \text{ALCUOI}, \text{ALCQUOI}, \text{ALCUOSelf}, \text{ALCUOISelf}$, and $\text{ALCQUOISelf}$ are closed under substitutions.

Theorem: The description logics $\text{ALCQUO}$ and $\text{ALCQUOSelf}$ are not closed under substitutions.
We define $wp(strat, Q)$ the weakest precondition for a strategy $strat$ and a formula $Q$.

- $wp(s_0; s_1, Q) = wp(s_0, wp(s_1, Q))$
- $wp(s_0 \oplus s_1, Q) = wp(s_0, Q) \land wp(s_1, Q)$
- $wp(\rho, Q) = App(\rho) \Rightarrow Q[a_n]...[a_0]$ where $\rho$’s right-hand side is $a_0; ...; a_n$
Generating Weakest Preconditions

We define $wp(strat, Q)$ the weakest precondition for a strategy $strat$ and a formula $Q$.

$$wp(\rho, Q) = App(\rho) \Rightarrow Q[a_n]...[a_0]$$

**Definition: Application Condition**

Given a rule $\rho$, the *application condition* $App(\rho)$ is a formula such that a graph $G \models App(\rho)$ iff there exists a match between the left-hand side of $\rho$ and $G$.
Generating Weakest Preconditions

We define $wp(strat, Q)$ the weakest precondition for a strategy $strat$ and a formula $Q$.

- $wp(a, Q) = Q[a]$
- $wp(\epsilon, Q) = Q$
- $wp(a; \alpha, Q) = wp(a, wp(\alpha, Q))$
- $wp(s_0; s_1, Q) = wp(s_0, wp(s_1, Q))$
- $wp(s_0 \oplus s_1, Q) = wp(s_0, Q) \land wp(s_1, Q)$
- $wp(\rho, Q) = App(\rho) \Rightarrow Q[a_n]...[a_0]$
Generating Weakest Preconditions

$\text{wp}(\text{strat}, Q)$ computes the weakest precondition for a strategy $\text{strat}$ and a formula $Q$.

- $\text{wp}(a, Q) = Q[a]$
- $\text{wp}(\epsilon, Q) = Q$
- $\text{wp}(a; \alpha, Q) = \text{wp}(a, \text{wp}(\alpha, Q))$
- $\text{wp}(s_0; s_1, Q) = \text{wp}(s_0, \text{wp}(s_1, Q))$
- $\text{wp}(s_0 \oplus s_1, Q) = \text{wp}(s_0, Q) \land \text{wp}(s_1, Q)$
- $\text{wp}(\rho, Q) = \text{App}(\rho) \Rightarrow Q[a_n]...[a_0]$
- $\text{wp}(s^*, Q) = \text{inv}_s$
Verification Conditions

\begin{itemize}
    \item $\text{vc} (\rho, Q) = \top$
    \item $\text{vc} (s_0; s_1, Q) = \text{vc} (s_0, \text{wp}(s_1, Q)) \land \text{vc} (s_1, Q)$
    \item $\text{vc} (s_0 \oplus s_1, Q) = \text{vc} (s_0, Q) \land \text{vc} (s_1, Q)$
    \item $\text{vc} (s^*, Q) =$
        \((\text{inv}_s \land \neg \text{App}(s) \Rightarrow Q) \land (\text{inv}_s \land \text{App}(s) \Rightarrow \text{wp}(s, \text{inv}_s)) \land \text{vc}(s, \text{inv}_s)\)
Let \( \text{spec} = (\text{Pre}, \text{strat}, \text{Post}) \) be a specification. We call correctness formula the formula \( \text{correct}(\text{spec}) = (\text{Pre} \Rightarrow \text{wp}(\text{strat}, \text{Post})) \land \text{vc}(\text{strat}, \text{Post}). \)

**Theorem:**
If \( \text{correct}(\text{spec}) \) is valid, then for all graphs \( G, G' \) such that \( G \rightarrow_{\text{strat}} G', G \models \text{Pre} \) implies \( G' \models \text{Post}. \)
Decidability of the verification

Theorem:
Let \( spec = (\text{Pre}, \text{strat}, \text{Post}) \) be a specification using one of the following DL logics \( \text{ALCUO}, \text{ALCUOI}, \text{ALCQUOI}, \text{ALCUOSelf}, \text{ALCUOISelf}, \text{ALCQUOISelf} \). Then, the correctness of \( spec \) is decidable.

Other considered decidable logics

- Extension of the dynamic logic PDL: C2PDL
- First-order Logic : fragments \( \exists^* \forall^* \) and \( C^2 \)
Conclusion

- Conferences and workshops: ICGT, ICMT, GCM, etc.
- Various Algebraic Approaches: DPO, SPO, SqPO, AGREE, PBPO, etc.
- Various Implementations: AGG, GROOVE, GP, PORGY, PROGRES, etc.
- General Framework: (Weak) Adhesive (HLR) Categories
- Other issues: Parallelism, Verification Techniques, Termination...