Game Theory
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Lecture 4

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Lecture 2-3 recap

• Proved existence of pure strategy Nash equilibrium in games with compact convex action sets and continuous concave utilities
• Defined mixed strategy Nash equilibrium
• Proved existence of mixed strategy Nash equilibrium in finite games
• Discussed computation and interpretation of mixed strategies Nash equilibrium

→ Nash equilibrium is not the only solution concept
→ Today: Another solution concept: evolutionary stable strategies
Outline

- Evolutionary stable strategies
Evolutionary game theory

• Game theory ↔ evolutionary biology
• Idea:
  – Relate strategies to phenotypes of genes
  – Relate payoffs to genetic fitness
  – Strategies that do well “grow”, those that obtain lower payoffs “die out”
• Important note:
  – Strategies are **hardwired**, they are not chosen by players
• Assumptions:
  – Within species competition: no mixture of population
Examples

- Using game theory to understand population dynamics
  - Evolution of species
  - Groups of lions deciding whether to attack in group an antelope
  - Ants deciding to respond to an attack of a spider
  - TCP variants, P2P applications

- Using evolution to interpret economic actions
  - Firms in a competitive market
  - Firms are bounded, they can’t compute the best response, but have rules of thumbs and adopt hardwired (consistent) strategies
  - Survival of the fittest == rise of firms with low costs and high profits
A simple model

- Assume simple game: two-player symmetric
- Assume **random tournaments**
  - Large population of individuals with hardwired strategies, pick two individuals at random and make them play the symmetric game
  - The player adopting the strategy yielding higher payoff will survive (and eventually gain new elements) whereas the player who “lost” the game will “die out”

- Start with entire population playing strategy $s$
- Then introduce a **mutation**: a **small** group of individuals start playing strategy $s'$
- Question: will the mutants survive and grow or die out?
A simple example (1)

• Have you already seen this game?
• Examples:
  – Lions hunting in a cooperative group
  – Ants defending the nest in a cooperative group
• Question: *is cooperation evolutionary stable?*
A simple example (2)

Player strategy hardwired $\rightarrow$ C

"Spatial Game"

All players are cooperative and get a payoff of 2

What happens with a mutation?
A simple example (3)

Focus your attention on this random “tournament”:

• Cooperating player will obtain a payoff of 0
• Defecting player will obtain a payoff of 3

Survival of the fittest: D wins over C
A simple example (4)

Player strategy hardwired → C
Player strategy hardwired → D
A simple example (5)
A simple example (6)

A small initial mutation is rapidly expanding instead of dying out.

Eventually, C will die out.

→ Conclusion: C is not ES

Remark: we have assumed asexual reproduction and no gene redistribution.
**Definition 1: Evolutionary stable strategy**

In a symmetric 2-player game, the pure strategy \( \hat{s} \) is ES (in pure strategies) if there exists \( \varepsilon_0 > 0 \) such that:

\[
(1 - \varepsilon)[u(\hat{s}, \hat{s})] + \varepsilon[u(\hat{s}, s')] > (1 - \varepsilon)[u(s', \hat{s})] + \varepsilon[u(s', s')]
\]

for all possible deviations \( s' \) and for all mutation sizes \( \varepsilon < \varepsilon_0 \).
ES strategies in the simple example

• Is cooperation ES?
  C vs. [(1-ε)C + εD] → (1-ε)2 + ε0 = 2(1-ε)
  D vs. [(1-ε)C + εD] → (1-ε)3 + ε1 = 3(1-ε)+ ε

→ C is not ES because the average payoff to C is lower than the average payoff to D

→ A strictly dominated is never Evolutionarily Stable
  – The strictly dominant strategy will be a successful mutation
ES strategies in the simple example

• Is defection ES?

D vs. \([\varepsilon C + (1-\varepsilon)D]\) \(\to\) \((1-\varepsilon)1 + \varepsilon3 = (1-\varepsilon)+3\varepsilon\)

C vs. \([\varepsilon C + (1-\varepsilon)D]\) \(\to\) \((1-\varepsilon)0 + \varepsilon2 = 2\varepsilon\)

\((1-\varepsilon)+3 > 2\varepsilon\)

\(\Rightarrow\) D is ES: any mutation from D gets wiped out!
Another example (1)

- 2-players symmetric game with 3 strategies
- Is “c” ES? $c$ vs. $[(1-\varepsilon)c + \varepsilon b] \rightarrow (1-\varepsilon) 0 + \varepsilon 1 = \varepsilon$
  $b$ vs. $[(1-\varepsilon)c + \varepsilon b] \rightarrow (1-\varepsilon) 1 + \varepsilon 0 = 1- \varepsilon > \varepsilon$

$\Rightarrow$ “c” is not evolutionary stable, as “b” can invade it

- Note: “b”, the invader, is itself not ES!
  - It is not necessarily true that an invading strategy must itself be ES
  - But it still avoids dying out completely (grows to 50% here)
Another example (3)

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
<td>a</td>
<td>2,2</td>
<td>0,0</td>
<td>0,0</td>
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<tr>
<td>b</td>
<td>0,0</td>
<td>0,0</td>
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<tr>
<td>c</td>
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- Is (c,c) a NE?
Observation

• If \( s \) is not **Nash** (that is \((s, s)\) is not a NE), then \( s \) is **not evolutionary stable** (ES)

Equivalently:

• If \( s \) is ES, then \((s, s)\) is a NE

• Question: is the opposite true? That is:
  – If \((s, s)\) is a NE, then \( s \) is ES
Yet another example (1)

• NE of this game: \((a,a)\) and \((b,b)\)
• Is \(b\) ES? \(b \rightarrow 0\)
  \(a \rightarrow (1-\varepsilon) 0 + \varepsilon 1 = \varepsilon > 0\)

\(\Rightarrow (b,b)\) is a NE, but it is not ES!
• This relates to the idea of a weak NE

\(\Rightarrow \) If \((s,s)\) is a **strict NE** then \(s\) is ES
Definition: **Strict Nash equilibrium**

A strategy profile \((s_1^*, s_2^*,..., s_N^*)\) is a strict Nash Equilibrium if, for each player \(i\),

\[
u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)\quad \text{for all } s_i \neq s_i^*
\]

- Weak NE: the inequality is an equality for at least one alternative strategy

- Strict NE is sufficient but not necessary for ES
ESS Definition 2

Definition 2: Evolutionary stable strategy

In a symmetric 2-player game, the pure strategy $\hat{s}$ is ES (in pure strategies) if:

A) $(\hat{s}, \hat{s})$ is a symmetric Nash Equilibrium
   
   $u(\hat{s}, \hat{s}) \geq u(s', \hat{s}) \quad \forall s'$

AND

B) if $u(\hat{s}, \hat{s}) = u(s', \hat{s})$ then

   $u(\hat{s}, s') > u(s', s')$
Link between definitions 1 and 2

<table>
<thead>
<tr>
<th>Theorem</th>
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<tbody>
<tr>
<td>Definition 1 ⇔ Definition 2</td>
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• Proof sketch:
Recap: checking for ES strategies

• We have seen a definition that connects Evolutionary Stability to Nash Equilibrium
• By def 2, to check that \( \hat{s} \) is ES, we need to do:

  – First check if \((\hat{s}, \hat{s})\) is a **symmetric** Nash Equilibrium
  – If it is a **strict** NE, we’re done
  – Otherwise, we need to compare how \( \hat{s} \) performs against a mutation, and how a mutation performs against a mutation
  – If \( \hat{s} \) performs better, then we’re done
Example: Is “a” evolutionary stable?

<table>
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<tr>
<th></th>
<th>Player 2</th>
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<tbody>
<tr>
<td>a</td>
<td>1,1</td>
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<td>1,1</td>
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<table>
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<tr>
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</thead>
<tbody>
<tr>
<td>a</td>
<td>1,1</td>
</tr>
<tr>
<td>ε</td>
<td>1-ε</td>
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</table>

• Is (a, a) a NE? Is it strict?
• Is “a” evolutionary stable?
Evolution of social convention

• Evolution is often applied to social sciences
• Let’s have a look at how driving to the left or right hand side of the road might evolve

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• What are the NE? are they strict? What are the ESS?

• Conclusion: we can have several ESS
  – They need not be equally good
The game of Chicken

- This is a **symmetric coordination game**
- Biology interpretation:
  - “a” : individuals that are aggressive
  - “b” : individuals that are non-aggressive
- What are the pure strategy NE?
  - They are not symmetric $\rightarrow$ no candidate for ESS

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</table>
The game of Chicken: mixed strategy

NE

\[
\begin{array}{cc}
& a & b \\
a & 0,0 & 2,1 \\
b & 1,2 & 0,0 \\
\end{array}
\]

• What’s the mixed strategy NE of this game?
  – Mixed strategy NE = [ (2/3, 1/3) , (2/3 , 1/3) ]
  ➔ This is a symmetric Nash Equilibrium

➔ Interpretation: there is an equilibrium in which 2/3 of the genes are aggressive and 1/3 are non-aggressive

• Is it a strict Nash equilibrium?
• Is it an ESS?
Remark

• A mixed-strategy Nash equilibrium (with a support of at least 2 actions for one of the players) can never be a strict Nash equilibrium

• The definition of ESS is the same!
### ESS Definition 2bis

**Definition 2: Evolutionary stable strategy**

In a symmetric 2-player game, the mixed strategy $\hat{s}$ is ES (in mixed strategies) if:

- **A)** $(\hat{s}, \hat{s})$ is a symmetric Nash Equilibrium
  
  $$u(\hat{s}, \hat{s}) \geq u(s', \hat{s}) \quad \forall s'$$

- **AND**

- **B)** if $u(\hat{s}, \hat{s}) = u(s', \hat{s})$ then
  
  $$u(\hat{s}, s') > u(s', s')$$
The game of Chicken: ESS

- Mixed strategy NE = [(2/3, 1/3), (2/3, 1/3)].
- Is it an ESS? we need to check for all possible mixed mutations s': \( u(\hat{s}, s') > u(s', s') \) \( \forall s' \neq \hat{s} \)
- Yes, it is (do it at home!)

- In many cases that arise in nature, the only equilibrium is a mixed equilibrium
  - It could mean that the gene itself is randomizing, which is plausible
  - It could be that there are actually two types surviving in the population (cf. our interpretation of mixed strategies)
Hawks and doves

Dove

Hawk
The Hawks and Dove game (1)

- More general game of aggression vs. non-aggression
  - The prize is food, and its value is $v > 0$
  - There’s a cost for fighting, which is $c > 0$

- Note: we’re still in the context of **within spices competition**
  - So it’s not a battle against two different animals, hawks and doves, we talk about strategies
    - “Act dovish vs. act hawkish”

- What are the ESS? How do they change with $c$, $v$?
The Hawks and Dove game (2)

<table>
<thead>
<tr>
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<th>H</th>
<th>D</th>
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<tbody>
<tr>
<td>H</td>
<td>(v-c)/2, (v-c)/2</td>
<td>v, 0</td>
</tr>
<tr>
<td>D</td>
<td>0, v</td>
<td>v/2, v/2</td>
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- Can we have a ES population of doves?
- Is (D,D) a NE?
  - No, hence “D” is not ESS
  - Indeed, a mutation of hawks against doves would be profitable in that it would obtain a payoff of v
The Hawks and Dove game (3)

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<td>0, v</td>
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• Can we have a ES population of Hawks?
• Is (H,H) a NE? It depends: it is a symmetric NE if (v-c)/2 ≥ 0

• Case 1: v>c → (H,H) is a **strict** NE → “H” is ESS
• Case 2: v=c → (v-c)/2 = 0 → u(H,H) = u(D,H) -- (H, H) is a weak NE
  – Is u(H,D) = v larger than u(D,D) = v/2? Yes → “H” is ESS

⇒ H is ESS if v ≥ c
• If the prize is high and the cost for fighting is low, then you’ll see fights arising in nature
The Hawks and Dove game (4)

<table>
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<tbody>
<tr>
<td>H</td>
<td>(v-c)/2, (v-c)/2</td>
<td>v,0</td>
</tr>
<tr>
<td>D</td>
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- What if c > v?
  - “H” is not ESS and “D” is not ESS (they are not NE)
- Step 1: find a mixed NE

- Step 2: verify the ESS condition
The Hawks and Dove game: results

• In case $v < c$ we have an evolutionarily stable state in which we have $v/c$ hawks
  1. As $v \uparrow$ we will have more hawks in ESS
  2. As $c \uparrow$ we will have more more doves in ESS

• By measuring the proportion of H and D, we can get the value of $v/c$

• Payoff: $E[u(D, \hat{s})] = E[u(H, \hat{s})] = 0 \frac{v}{c} + \left(1 - \frac{v}{c}\right) \frac{v}{2}$
One last example (1)

- Assume $1 < v < 2$
  - ~ Rock, paper, scissors
- Only NE: $\hat{s} = (1/3, 1/3, 1/3)$ – mixed, not strict
- Is it an ESS?
  - Suppose $s' = R$
  - $u(\hat{s}, R) = (1 + v)/3 < 1$
  - $u(R, R) = 1$

- Conclusion: Not all games have an ESS!