

The Heterogeneous Colonel Blotto Game

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Abstract—The Colonel Blotto game, proposed by Borel in 1921, is a fundamental model of strategic resource allocation. Two players allocate an exogenously given amount of resources to a fixed number of battlefields with given values. Each battlefield is then won by the player who allocated more resources to it, and each player maximizes the aggregate value of battlefields he wins. This game allows modeling many practical problems of resource allocation in various strategic settings ranging from international war to competition for attention in social networks; it is particularly useful in security to model the allocation of defense resources on different potential targets. The scope of applications, however, has been limited by the lack of solutions of the game in realistic scenarios. Indeed, despite its apparent simplicity, the Colonel Blotto game is very intricate and it remains unsolved in the case with asymmetric players (i.e., with different resources) and with an arbitrary number of battlefields that can have different values.

In this paper, we propose a solution of the heterogeneous Colonel Blotto game with asymmetric players and heterogeneous battlefield values, under the assumption that there is a sufficient number of battlefields of each possible value relative to the players’ resources asymmetry. In particular, our assumption implies that there must be at least three battlefields of each possible value. Then, we characterize the unique equilibrium payoffs and univariate marginal distributions, along with the proof that there exist n -variate joint distributions with such marginals. Our results expand the scope of potential applications of the Colonel Blotto game, and mark a new step towards a complete solution of the game.

I. INTRODUCTION

The Colonel Blotto game was introduced in 1921, by Borel to model “*the art of war or of economic and financial speculation*”, [1]. There exist many versions of this classical game, which were developed overtime by various research communities to target different applications ranging from modeling international conflicts to studying social interactions on Facebook. The Colonel Blotto games (or ‘Blotto games’ for short) model scenarios where strategic players with exogenously given resource levels compete for multiple battlefields with given values.

In its basic variant, the Colonel Blotto game is a game between two players who simultaneously allocate their ex-

ogenously given resources across a finite number of battlefields with given values. Players’ resources are assumed to have no outside value. A player wins a battlefield when the amount of resources allocated to this battlefield is higher than the said amount for his opponent. Each player’s objective is to maximize his cumulative winnings, i.e., the total value of the won battlefields. Players strategies and payoffs are common knowledge. Despite its deceptive simplicity, this Blotto game is intricate. It stood unresolved for several decades even for the simple case of symmetric players with identical resources and a fixed number of identical (i.e., of equal value) battlefields.

A. Literature review

The original 1921 Colonel Blotto game [1] was solved in 1938 for the case of two players with symmetric resources and three battlefields by Borel and Ville, [2]. In 1950, Gross and Wagner [3] extended [2] to the game with identical player resources and an arbitrary number of battlefields; they also derived an equilibrium for the case with two battlefields and asymmetric (unequal) player resources. The case of asymmetric player resources and an arbitrary number of battlefields remained unsolved until 2006. Only then, Roberson [4] completely characterized the equilibria of a two-player Blotto game with any given number of identical battlefields and asymmetric player resources. The latter paper builds on the results for all-pay auctions established in [5], [6], and uses the results of Sklar [7] and Schweizer and Sklar [8] on copulas to demonstrate the existence of an n -variate distribution corresponding to the equilibrium set of n univariate marginal distribution functions.

The literature considers numerous versions of Blotto-type games. These games could differ by procedures according to which the winner of each specific battlefield is determined, by the objective of the entire contest, and by the timing of the game. The most well studied objective is the plurality objective (i.e., maximizing the aggregate value of battlefields won); in this paper we assume exactly that. Another popular objective is the majority objective – in this case, each player maximizes the probability of winning a majority of the component contests.¹ The original Blotto model assumes that the players move simultaneously. A number of recent papers considered consider a simplified model, with sequential players moves, see for example [9], [10]. A comprehensive recent review of Blotto-type allocation games is provided in [11].

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¹US presidential elections is an example of such a contest.

Following the fundamental paper [4], many interesting theoretical extensions were proposed, and numerous papers targeting specific application domains emerged; see for example [12], [13], [14], [15], [16], [17], [18] among many others. In computer security for instance, the Blotto game can be applied to model the allocation of monitoring resources to different domains of a system. There have also been experimental studies for various applications domains, such as [19], [20], [21]. One of the first experimental studies that looked into network infrastructures is [22]. Another interesting experimental paper is [23] where the authors study social interactions using a Facebook application called “Project Waterloo”, which allows users to invite both friends and strangers to play Colonel Blotto against them.

The applications of the Colonel Blotto game, however, have been restricted by the lack of a solution for the case of asymmetric players and heterogeneous battlefields. This heterogeneous Blotto game was solved in [3] for the case of two battlefields. It was recently approached for more than two battlefields in [24]. There, the authors consider a more general nonzero-sum case where players could have asymmetric battlefield valuations. They address only, however, the cases where a pure Nash equilibrium exists. They construct an algorithm to find such equilibria, and characterize the situations in which heterogeneous nonzero-sum Blotto games have a pure strategy equilibria. That paper represents a growing literature that extends a standard zero-sum Blotto setup to the case of players with heterogeneous preferences, and thus non zero-sum games. Our paper is complementary to [24]. Our primary focus is on solving the heterogeneous zero-sum Blotto games in cases where no pure strategy equilibria exists, which happens when players resources are not too asymmetric. Then, under a condition on the number of battlefields of each specific value relative to the players resources asymmetry, we fully characterize mixed strategy equilibria of such heterogeneous Blotto games.

B. Contributions

The Colonel Blotto game is a canonical model of strategic resource allocation for players with multi-dimensional objectives. In this paper, we provide a solution of this fundamental problem for the case in which the battlefields could differ in value, and players could differ in the amount of available resources. We work under the assumption that there is a sufficient number of battlefields of each possible value relative to the players’ resources asymmetry (see condition (1) in Theorem 1). In particular, our assumption implies that there must be at least three battlefields of each possible value. Then, our results completely characterize the equilibrium. We present the unique equilibrium payoffs and the unique equilibrium univariate marginal distributions, along with the proof that there exist n -variate joint distributions with such marginals. This solution of the heterogeneous Blotto game opens up many potentially applications where battlefields naturally have different values. We identify novel practical applications where our results are useful. In addition, we anticipate that our solution of heterogeneous Blotto game is a

necessary component in modeling Blotto-type interactions in networked environments with heterogeneous network nodes (battlefields).

More specifically, Theorem 1 establishes unique marginal distributions with which any equilibrium resource allocation has to comply. The existence of some n -variate distribution that satisfies these marginals and also player resource constraints is merely hypothesized. The result of Theorem 1 importantly relies on constructing a one-to-one correspondence between the solution of our Blotto game and the solution of n independent all-pay auctions, with the latter being already established in the literature [5], [6].

Next, in Theorem 3 we establish the existence of an n -variate distribution that was assumed to exist for the proof of Theorem 1. The existence of such an n -variate distribution with the required properties is proven by a construction in two steps. First, each group of battlefields with the same value is deterministically allocated an amount of resources proportional to its aggregate value. Second, within each group, the allocation to battlefields is done with randomization as in Roberson [4]. The latter step is possible thanks to our imposition of having at least three battlefields with the same value.

C. Outline of the paper

The reminder of the paper is organized as follows. In Section II, we introduce the notation and define the heterogeneous Colonel Blotto game. In Section III, we formulate our main results, discuss them and briefly outline the main steps of the proof. The technical details of the longer proofs are presented in Sections IV and V. Lastly, in Section VI we discuss the results and outline the plans for extensions.

II. PROBLEM FORMULATION

We consider a Colonel Blotto game between two players, each of whom has a fixed amount of resources to allocate between a given number of battlefields. Each battlefield has a specific value. Players simultaneously allocate their resources between the battlefields. We assume that player resources have no outside value. The parameters of the game (players’ resources, number and values of the battlefields), players’ actions spaces and payoffs are common knowledge. A player who allocated a higher amount of resources to a specific battlefield wins this battlefield. The payoff of each player is equal to the sum of the values of the battlefields that he won. Each player’s objective is to maximize his payoff.

We now introduce the notation and formally define the game. Whenever possible, our notation follows the notation from Roberson [4], which we extend by allowing battlefields of different values. When all battlefields have equal value, our setting reverts to [4].

We denote by A and B the two players and by $X_A \in \mathbb{R}_+$ and $X_B \in \mathbb{R}_+$ their respective resources. Without loss of generality, we assume that $X_A \leq X_B$. Let n be an integer denoting the number of battlefields. Each battlefield $j \in \{1, \dots, n\}$ is endowed with a value $v^j \in \mathbb{R}_+$. We denote

by $\mathbf{v} = (v^1, \dots, v^n)$ the vector of battlefield values, and by $V = \sum_{j=1}^n v^j$ the aggregate value of all battlefields.

Players choose how to allocate their resources between the battlefields, i.e., each player chooses how to distribute his (given) resources between the battlefields. Formally, a pure strategy of player $p \in \{A, B\}$ is a vector $\mathbf{x}_p = (x_p^j)_{j \in \{1, \dots, n\}} \in \mathbb{R}_+^n$ satisfying the budget constraint $\sum_{j=1}^n x_p^j \leq X_p$, where $x_p^j \in \mathbb{R}_+$ denotes the amount of resources allocated to battlefield j by player p . We let S_p denote the set of pure strategies for player p :

$$S_p = \left\{ \mathbf{x}_p \in \mathbb{R}_+^n : \sum_{j=1}^n x_p^j \leq X_p \right\}, \quad (p \in \{A, B\}).$$

For each battlefield $j \in \{1, \dots, n\}$, the player who dedicates the highest amount of resources wins this entire battlefield. Without loss of generality, we assume that in case of a tie, player B (the player with higher total amount of resources) wins the battlefield. Hence, for each battlefield $j \in \{1, \dots, n\}$, if $x_A^j > x_B^j$ then player A wins the battlefield, and if $x_A^j \leq x_B^j$ then player B wins it. For each player, the payoff in the game equals to the sum of the values of the battlefields that he wins.

Let $\mathfrak{B}(X_A, X_B, \mathbf{v})$ denote the above presented Blotto game. The game \mathfrak{B} is a one-shot game in which players A and B simultaneously choose their allocation of forces to the battlefields to maximize the total value of the battlefields they win. The game \mathfrak{B} is a complete information game. The parameters of the game (X_A, X_B, n and \mathbf{v}), players' action spaces and objectives are common knowledge.

In most cases of interest, that is when for each player the expected payoff is strictly positive, no pure strategy Nash equilibrium exists. We therefore will focus on mixed strategy equilibria. A mixed strategy for player $p \in \{A, B\}$ is an n -variate distribution $F_p : \mathbb{R}_+^n \rightarrow [0, 1]$ whose support is contained in S_p . For a given n -variate distribution, we denote by $F_p^j : \mathbb{R}_+ \rightarrow [0, 1]$ univariate marginal distribution of resources allocated by player p to battlefield j .

This problem was solved by Roberson [4] for the case of identical battlefields. He proved the existence of a Nash equilibrium by constructing a mixed-strategy equilibrium and characterized equilibrium marginals and payoffs, which turned out to be unique. In this paper, we solve a generalization of that problem to the case where the battlefields may have different values. As we will demonstrate in the next section, there exists a plethora of equilibria (in terms of the n -variate joint distributions), but all these equilibria have identical marginals and payoffs.

To prove our results, we will need to assume that, for each battlefield, there is at least two other battlefields with the same value, i.e., we can make groups of battlefields of the same value with at least three battlefields in each group. Moreover, we will also assume that each group of equal-value battlefields has "enough" battlefields compared to the players' resources asymmetry (see Theorem 1 below). To precisely state these conditions and establish our framework, we introduce the following formal definitions.

Let k denote the number of different battlefield values, and let $\{w_1, \dots, w_k\}$ denote the corresponding set of unique battlefield values. Formally we have $w_{i_1} \neq w_{i_2}$ for all $i_1 \neq i_2$ and, for all $j \in \{1, \dots, n\}$ there exists $i \in \{1, \dots, k\}$ such that $v^j = w_i$. Note that $n \geq 3k$ since we assume that there are more than two battlefields of each unique value. For $i \in \{1, \dots, k\}$, define $C(i)$ as the set of battlefields with value w_i , i.e., $C(i) = \{j \in \{1, \dots, n\} : v^j = w_i\}$. Define, for each $i \in \{1, \dots, k\}$, $n_i = \#C(i)$ the number of battlefields of value w_i (again, $n_i \geq 2$ for all i) and $V_i = \sum_{j \in C(i)} v^j = n_i w_i$ the aggregate value of all battlefields of value w_i . Note that $\sum_{i=1}^k V_i = V$.

Throughout the paper, quantities with exponent j refer to battlefield j (e.g., v^j the value of battlefield j); whereas quantities with index i refer to the group of battlefield i (e.g., w_i the value of battlefields in group i and n_i the number of battlefields in group i).

III. MAIN RESULTS

We start by characterizing the unique equilibrium marginal distributions for the game $\mathfrak{B}(X_A, X_B, \mathbf{v})$:

Theorem 1: Assume that, for all groups of battlefields $i \in \{1, \dots, k\}$, we have

$$\frac{2}{n_i} < \frac{X_A}{X_B} \leq 1. \quad (1)$$

Then, in equilibrium, each player allocates resources with the following unique univariate marginal distribution functions $\forall j \in \{1, \dots, n\}$:

(i) For player A:

$$F_A^j(x) = \left(1 - \frac{X_A}{X_B}\right) + \frac{x}{\frac{2v^j}{V} X_B} \left(\frac{X_A}{X_B}\right), \quad x \in [0, \frac{2v^j}{V} X_B]; \quad (2a)$$

(ii) For player B:

$$F_B^j(x) = \frac{x}{\frac{2v^j}{V} X_B}, \quad x \in [0, \frac{2v^j}{V} X_B]. \quad (2b)$$

To improve the exposition, the detailed proof is relegated to Section IV. Below we briefly summarize the steps of the proof. The proof works by establishing a one-to-one correspondence between the solution of the Blotto game and the solution of n independent all-pay auctions, with the later established in [5], [6].

Theorem 1 (and the subsequent results) requires that condition (1) holds. First note that it implies that $n_i \geq 3$, i.e., each group of battlefields has at least three battlefields of the same value. This ensures that we will be able to construct an n -variate distribution with the marginals in Theorem 1 (see Theorem 3 below and the discussion that follows); which is necessary to prove Theorem 1. Condition (1) also restricts the disparity of players resources. However, even for a large asymmetry, this condition will be satisfied as soon as the number of battlefields is large enough in each group. We note that such an assumption is often made in the literature to restrict the complexity of the analysis, for instance in [25]. The cases in which condition (1) is not met can be studied separately but are of limited practical interest.

Under the condition of Theorem 1, we have unique equilibrium marginals. These marginals are uniform, as in the game with identical battlefield values, but now the marginal's support is proportional to the battlefields value. Theorem 1 allows us to obtain the equilibrium player payoffs:

Corollary 2: Under condition (1) of Theorem 1, in equilibrium, player A and B expected payoffs are $V \frac{X_A}{2X_B}$ and $V \left(1 - \frac{X_A}{2X_B}\right)$, respectively.

Proof: By direct computation, the expected payoff of player A with the equilibrium marginals of Theorem 1 is

$$\begin{aligned} & V \int_0^{\frac{2v^j}{V} X_B} F_B^j(x) dF_A^j \\ &= V \cdot \int_0^{\frac{2v^j}{V} X_B} \frac{x}{\frac{2v^j}{V} X_B} \cdot \frac{1}{\frac{2v^j}{V} X_B} \left(\frac{X_A}{X_B}\right) dx \\ &= V \cdot \frac{X_A}{X_B} \cdot \frac{1}{2} = V \frac{X_A}{2X_B}. \end{aligned}$$

The computation of expected payoff for player B is similar. Alternatively, his payoff can be inferred from the fact that the sum of player payoffs is V . ■

Remarkably, Corollary 2 yields equilibrium payoffs identical the payoffs in the game with equal battlefield values.

Theorem 1 describes only the marginal distributions. So far, we merely hypothesized the existence of some n -variate distribution with such marginals that respects player resource constraints. Next, we will establish the existence of such n -variate distribution.

Theorem 3: Under condition (1) of Theorem 1, for each player $p \in \{A, B\}$, there exists an n -variate distribution with support contained in S_p such that the marginals are given by (2a) - (2b) for all battlefields $j \in \{1, \dots, n\}$.

The proof of Theorem 3 is presented in Section V. The proof is done by constructing an n -variate distribution with the correct marginals that respects the players budget constraints. Roughly speaking, the construction consists of two steps. First, we make a deterministic allocation to each group of battlefields with the same value. The amount of allocated resources is proportional to the aggregate value of the group. Second, within each group the randomization is done as in Roberson [4]. This is possible because, by our assumption, each group has at least three battlefields. Our construction allows to obtain an equilibrium n -variate distribution with the correct marginals respecting the budget constraints. Still, there may exist other solutions that randomize the global resource allocation between the different groups of battlefields with a common value.

Our results are consistent with the recent paper by Kovenock and Roberson [25] on coalitional Blotto games. In [25], one player (A), is fighting two simultaneous disjoint Blotto games, \mathfrak{B}_1 and \mathfrak{B}_2 , and each of the games has identical battlefields. Nominally, these games are completely separate; still, they are related because player A has to allocate the common resources between them. In contrast with player A , each of his opponents $i \in \{1, 2\}$ in the games \mathfrak{B}_i is engaged in a single Blotto game only. The authors of

[25] consider a two-stage game, in which at the first stage (ex-ante), players 1 and 2 can form an alliance and transfer resources to each other. At the second stage of the game, each player i confronts player A in the game \mathfrak{B}_i with resources updated via ex-ante transfer.

The primary focus of [25] is on non-cooperative case in the absence of commitment (no ex-post transfers between the players could be enforced); and the authors demonstrate that for a range of parameters, a positive ex-ante transfer occurs in equilibrium even with no commitment. However, in Section 5, they also present a benchmark case of fully committed alliances. In that case, players 1 and 2 can make binding commitments about the ex post payoff allocation. With commitment, in equilibrium, players 1 and 2 maximize cumulative payoff – the payoff of a social planner fighting against player A . Note that the objective of such a social planner corresponds to player objective(s) in a heterogeneous Blotto game with the battlefields that combine the ones of \mathfrak{B}_1 and \mathfrak{B}_2 . The authors of [25] show that the optimal ex-ante transfer gives players 1 and 2 resources proportional to the total values of \mathfrak{B}_1 and \mathfrak{B}_2 respectively, which is consistent with our Theorem 1 and Corollary 2. Yet our results significantly differ from [25]. Firstly, we consider a simultaneous game rather than a two-stage game. Secondly, we demonstrate uniqueness, which cannot be derived from [25]. Lastly, our setting is more general, as it covers an arbitrary number of differently valued battlefields instead of two only.

IV. PROOF OF THEOREM 1

Our proof follows the same internal logic as the proof of Theorem 2 in [4],² which we revisit and extend to the case where battlefield values may differ. The proof of Theorem 1 is based on the results for all-pay auctions established in [5], [6], where the uniqueness of the equilibrium marginals is demonstrated. For completeness, below we briefly recall the relevant results.

The proof of Theorem 1 is done under an assumption that Theorem 3 holds, i.e., that it will be possible to construct a joint distribution with the aforementioned marginals. Theorem 3 is proved independently in Section V.

A. Preliminaries: results from the all-pay auctions literature

Consider a two-bidders all-pay auction with complete information. Let $\{A, B\}$ denote the set of bidders, and v_p denote the value of the object for bidder $p \in \{A, B\}$. By the rules of all-pay auctions, both bidders pay their bids regardless of the outcome, and upon winning the auction, the winning bidder receives the object which he values at v_p . Let F_p denote the distribution of bids of bidder p , and let x_p denote the bid chosen by bidder p . We will use the subscript $-p$ to indicate his opponent. The probability for bidder p to win the auction is $Pr(x_p \geq x_{-p}) = F_{-p}(x_p)$.

²See also [26] and [27] for errata to [4].

Then, we can state that in equilibrium, each bidder must choose his distribution F_p to solve

$$\max_{F_p} \int_0^\infty (v_p F_{-p}(x) - x) dF_p.$$

By applying the equilibrium characterization from [5], [6] to our setting, we infer that there exists a unique equilibrium, and it is given by

(i) if $v_p \geq v_{-p}$, then

$$F_p(x) = \frac{x}{v_{-p}}, \quad x \in [0, v_{-p}]; \quad (3a)$$

(ii) if $v_p \leq v_{-p}$, then

$$F_p(x) = \left(\frac{v_{-p} - v_p}{v_{-p}} \right) + \frac{x}{v_{-p}}, \quad x \in [0, v_p]. \quad (3b)$$

Next, let us consider n independent two-bidders all-pay auctions with complete information. Let each auction $j \in \{1, \dots, n\}$ has a value v_p^j for bidder $p \in \{A, B\}$. Let F_p^j denote the distribution of bids of bidder p in auction j . Each bidder chooses his distributions of bids to solve

$$\max_{\{F_p^j\}_{j \in \{1, \dots, n\}}} \sum_{j=1}^n \int_0^\infty (v_p^j F_{-p}^j(x) - x) dF_p^j. \quad (4)$$

Since the auctions are independent, in equilibrium the marginals are uniquely determined for each auction j by (3a) - (3b), with v_p and v_{-p} replaced respectively by v_p^j and v_{-p}^j .

B. Proof of Theorem 1

Armed with the results of previous section on all-pay auctions, we now proceed to proving Theorem 1. First, we show in the next lemma that the marginals in Theorem 1, together with the joint distributions in Theorem 3, indeed constitute a Nash equilibrium, i.e., that they are best responses to each other.

Lemma 4: Under the assumptions of Theorem 1, the n -variate distributions from Theorem 3 (that have the marginals given by (2a)-(2b)) form an equilibrium of $\mathfrak{B}(X_A, X_B, v)$.

Proof: Suppose that player B 's strategy has marginals (2b). Let $(x_A^j)_{j \in \{1, \dots, n\}} \in S_A$ be a pure strategy of player A . Then, his expected payoff is

$$\sum_{j=1}^n v^j \cdot F_B^j(x_A^j) \leq \sum_{j=1}^n v^j \cdot \frac{x_A^j}{\frac{2v^j}{V} X_B} \leq V \cdot \frac{X_A}{2X_B}; \quad (5)$$

where the first inequality uses the fact that

$$F_B^j(x_A^j) = \frac{x_A^j}{\frac{2v^j}{V} X_B} \text{ if } x_A^j \leq \frac{2v^j}{V} X_B,$$

and

$$F_B^j(x_A^j) = 1 < \frac{x_A^j}{\frac{2v^j}{V} X_B} \text{ if } x_A^j > \frac{2v^j}{V} X_B,$$

and the second inequality in (5) uses the fact that $\sum_{j=1}^n x_A^j \leq X_A$ by definition of S_A .

From direct computation (similar to the computation in the proof of Corollary 2), we obtain that $V \frac{X_A}{2X_B}$ is the payoff achieved with marginals (2a) against marginals (2b). We

infer that playing with the marginals (2a) for player A is a best response to the marginals (2b) for player B . A similar reasoning provides that playing with the marginals (2b) for player B is a best response to the marginals (2a) for player A . Hence, the marginals in (2a)-(2b) in Theorem 1 constitute a Nash equilibrium. \blacksquare

In the rest of the proof, we show that, if a profile is an equilibrium, it must have the form as given by Theorem 1. Let $p \in \{A, B\}$ denote a player in the game $\mathfrak{B}(X_A, X_B, v)$. For a given strategy P_{-p} of his opponent, with marginals $\{F_{-p}^j\}_{j \in \{1, \dots, n\}}$, player p select P_p to solve

$$\max_{P_p} \sum_{j=1}^n \int_0^\infty v^j F_{-p}^j(x) dF_p^j, \quad (6)$$

subject to the constraint that the support of P_p is contained in S_p . This is equivalent to solving the following constrained optimization problem:

$$\max_{\{F_p^j\}_{j \in \{1, \dots, n\}}} \sum_{j=1}^n \int_0^\infty v^j F_{-p}^j(x) dF_p^j, \quad (7)$$

subject to constraints **(C1)** and **(C2)**:

(C1) there exists an n -variate distribution with marginals $\{F_p^j\}_{j \in \{1, \dots, n\}}$ that puts positive weight only on allocations whose sum is below the sum of expected allocation on each battlefield, i.e.,

$$\text{Support}(P_p) \subseteq \left\{ \mathbf{x}_p \in \mathbb{R}_+^n : \sum_{j=1}^n x_p^j \leq \sum_{j=1}^n \int_0^\infty x dF_p^j \right\},$$

and

(C2) the sum of expected allocations on each battlefield is below X_p , i.e.,

$$\sum_{j=1}^n \int_0^\infty x dF_p^j \leq X_p.$$

Integrating condition **(C2)** through a Lagrange multiplier λ_p , player p 's problem becomes

$$\max_{\{F_p^j\}_{j \in \{1, \dots, n\}}} \sum_{j=1}^n \int_0^\infty (v^j F_{-p}^j(x) - \lambda_p x) dF_p^j + \lambda_p X_p, \quad (8)$$

subject to $\lambda_p \geq 0$ and **(C1)**.

From now on, we assume that we are at a Nash equilibrium. First observe that, at equilibrium, both players use their resource budget entirely, hence $\lambda_p > 0$ for each $p \in \{A, B\}$. Indeed, suppose that there exists an equilibrium such that player $-p$ does not use all of his resources. By Lemma 4, player p has payoff strictly larger than the one in Corollary 2. Indeed, if player $-p$ does not use all the resources, player p could obtain the payoff strictly higher than the one in Corollary 2 by playing the strategy of Theorem 3 that has the marginals of Theorem 1. Since the sum of expected payoffs of both players is a constant (it is equal to V), player's $-p$ payoff is strictly smaller than the one in Corollary 2. This contradicts the fact that we are at equilibrium because by Lemma 4, player $-p$ could obtain a payoff at least equal

to the one in Corollary 2 against any player p strategy by employing the strategy from Theorem 3 with marginals from Theorem 1. We conclude that, in equilibrium, both players use all of their resource.

Note that our reasoning above relies on Lemma 4 that demonstrates that the marginals in Theorem 1 constitute a Nash equilibrium. Thus, it relies on condition (1): without this condition the marginals are not well defined.³

Next, we re-write (8) as

$$\max_{\{F_p^j\}_{j \in \{1, \dots, n\}}} \lambda_p \sum_{j=1}^n \int_0^\infty \left(\frac{v^j}{\lambda_p} F_{-p}^j(x) - x \right) dF_p^j + \lambda_p X_p, \quad (9)$$

subject to $\lambda_p > 0$ and **(C1)**. The objective function (9) is the same as (4) (up to multiplicative and additive constants), if we replace v_p with v^j/λ_p . By a straightforward adaptation of Theorem 3, we know that **(C1)** is satisfied for the marginals given by (3a)-(3b) for any values v_p . Therefore, the equilibrium marginals must satisfy (3a)-(3b) by replacing v_p by v^j/λ_p , that is, for all $j \in \{1, \dots, n\}$:

(i) if $\lambda_p \leq \lambda_{-p}$, then

$$F_p^j(x) = \frac{x\lambda_{-p}}{v^j}, \quad x \in [0, \frac{v^j}{\lambda_{-p}}]; \quad (10a)$$

(ii) if $\lambda_p \geq \lambda_{-p}$, then

$$F_p^j(x) = \left(1 - \frac{\lambda_{-p}}{\lambda_p}\right) + \frac{x\lambda_{-p}}{v^j}, \quad x \in [0, \frac{v^j}{\lambda_p}]. \quad (10b)$$

Note that it is also possible, as in [4], to obtain this characterization by adapting straightforwardly each step from the proofs for all-pay auctions. Here we have chosen to apply the existing results from [5], [6] to shorten our exposition.

Knowing that there exists an n -variate distribution such that constraint **(C1)** is satisfied for the marginals (3a)-(3b) allowed us to conclude that the marginals must have the same structure as for n independent all-pay auctions. However, the set of n -variate distributions such that **(C1)** is satisfied is clearly a strict subset of the set of equilibrium n -variate distributions for the n all-pay auctions problem. Indeed, for instance, the independent n -variate distribution is a solution for the n all-pay auctions problem whereas it clearly does not satisfy **(C1)**.

To conclude the proof of Theorem 1, we show that:

Lemma 5: At equilibrium, Lagrange multipliers are uniquely determined by

(i) for player A:

$$\lambda_A = \frac{V}{2X_B} \quad (11a)$$

(ii) for player B:

$$\lambda_B = \frac{X_A \cdot V}{2X_B^2}. \quad (11b)$$

Proof: The Lagrange multipliers are determined by the resource budget constraints (in expectation computed from

the marginals). Recall that $X_A \leq X_B$, and consider two cases: $\lambda_A \geq \lambda_B$ and $\lambda_A < \lambda_B$.

First, assume $\lambda_A \geq \lambda_B$. Then the marginal of player B is given by (10a) and the budget constraint gives:

$$X_B = \sum_{j=1}^n \int_0^{\frac{v^j}{\lambda_A}} \frac{x\lambda_A}{v^j} dx = \sum_{j=1}^n \frac{v^j}{2\lambda_A},$$

which gives (11a). Similarly, the marginal of player A is given by (10b) and using the budget constraint provides:

$$X_A = \sum_{j=1}^n \int_0^{\frac{v^j}{\lambda_A}} \frac{x\lambda_B}{v^j} dx = \sum_{j=1}^n \frac{v^j\lambda_B}{2\lambda_A^2} = \frac{2\lambda_B X_B^2}{V},$$

which gives (11b).

Next, assume $\lambda_A < \lambda_B$. Then, by the same type of computation, we obtain $\lambda_A = \frac{X_B \cdot V}{2X_A^2}$, and $\lambda_B = \frac{V}{2X_A} = \lambda_A \cdot \frac{X_A}{X_B}$ which contradicts $\lambda_A < \lambda_B$ because $X_A \leq X_B$. This completes the proof of Lemma 5. ■

We combine (11a)-(11b) with (10a)-(10b) to obtain that at Nash equilibrium the marginals are uniquely determined by (2a)-(2b), which completes the proof of Theorem 1. ■

V. PROOF OF THEOREM 3

In the proof of Theorem 4 Roberson [4] provides a procedure to construct an n -variate distribution with the marginals as in Theorem 1 for the case where all battlefields have equal value, when there exists at least three battlefields. To prove Theorem 3, we apply the same procedure separately to each group of battlefields with equal value.

We construct the allocation for an arbitrary player $p \in \{A, B\}$. For each group of battlefields $i \in \{1, \dots, k\}$, allocate an amount $\frac{V_i}{V} X_p$ of resources to the battlefields in $C(i)$ according to the process described in the proof of Theorem 4 of [4] (i.e., use this process replacing X_p in [4] by $\frac{V_i}{V} X_p$). By Theorem 4 of [4], this process gives positive weight only to allocations that respect the budget constraint individually for each group of battlefields of the same value, i.e., such that $\sum_{j \in C(i)} x_p^j \leq \frac{V_i}{V} X_p$ for all $i \in \{1, \dots, k\}$. Therefore, it is clear that distribution of the global allocation obtained satisfies the global budget constraint, i.e., has support contained in S_p .

By Theorem 4 of [4], the marginal for a battlefield $j \in C(i)$ ($i \in \{1, \dots, k\}$) has the form of (2a)-(2b) with $\frac{2v^j}{V} X_B$ replaced by $\frac{2}{n_i} \frac{V_i}{V} X_B$. Since $\frac{2}{n_i} \frac{V_i}{V} = \frac{2}{n_i} \frac{n_i w_i}{V} = \frac{2v^j}{V}$ for all $j \in C(i)$, the marginals obtained coincide with the ones in Theorem 1. This concludes the proof of Theorem 3. ■

VI. CONCLUDING REMARKS

This paper examines the heterogeneous Colonel Blotto game: a resource allocation game between two players with given resources who fight over a fixed number of battlefields with arbitrary values with the objective of maximizing the aggregate value of battlefields won. We characterize unique equilibrium payoffs of the players, and their strategies to achieve them.

This paper generalizes prior literature which is restricted to either homogeneous battlefields or symmetric players.

³For the analysis of Blotto game with identical battlefields and highly asymmetric player resources see Theorems 3 and 5 in [4].

In particular, it generalizes Roberson [4] by allowing the battlefields to have different values (when all battlefields have equal values, our setting reverts to [4]). Then, our results allow extending known variants of the Blotto game to environments where the battlefields importantly differ in their values. For example, combining our results with [25] allows finding equilibria of multi-player Blotto-type games. Indeed, our results permit to model alliances between the players engaged in heterogeneous Blotto games against the common enemy. Also, combining our results with [14], where players could add the battlefields at an exogenous cost, permits to analyze heterogeneous Blotto games with exogenous dimensionality. In the future, we plan to extend the analysis to a multi-player version of heterogeneous Colonel Blotto game, and especially to allocative games of the networked players. Our solution of heterogeneous Colonel Blotto game is a necessary step towards solving Blotto-type problems in network environments, where the networked battlefields have arbitrary non-identical values.

Our results also expand the scope of potential applications of the Blotto games to novel practical areas where battlefields naturally have different values. In particular, we plan to investigate in the future applications to the defense of cyber-physical systems where different potential targets may have different values.

In our analysis, we have assumed that the number of battlefields of each value is large enough relative to the players asymmetry, and at least larger than three (condition (1) of Theorem 1). This implies that each group of battlefields of the same value is in the regime described by Theorem 2 of [4] where the payoff is linear in the resource. This regime is by far the most studied in the literature. Yet, we are currently investigating the extension of our results to cases where different groups of battlefields are in different regimes such as the non-linear regime of Theorem 3 of [4]; as well as to cases with possibly less than three battlefields of a given value.

In our analysis of the Colonel Blotto game (as in most prior formulations of this game), we have assumed that both players have the same valuation of each battlefield. However, players may differ in their valuation of the battlefields. For example, consider cases of industrial espionage in which attackers exploit vulnerabilities in computer software to gain business advantage by learning competitor's trade secrets or cause disruptions in competitor's network operations. The IT-based attacks utilize malware and spyware to gain access to new technologies, and the competitors' business plans; also, the DoDs attacks are employed to sabotage the competitors' QoS and possibly the ability to serve users. In such cases, defenders' losses do not necessarily translate one to one into attackers' gains. Still, as long as players know each other valuations, our results are easily modifiable to solve such games. Indeed, our analysis extends straightforwardly to this case by using an analogy to the all-pay auctions in which players have different valuations of an object.

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