# Natural Deduction: quantifiers, copy and equality

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# Motivation

## Propositional case

There are algorithms to decide whether a given formula is valid or not.

## First-order case

There is no algorithm to decide whether a given formula is valid or not.

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### Propositional case

There are algorithms to decide whether a given formula is valid or not.

### First-order case

There is no algorithm to decide whether a given formula is valid or not.

If we assume the equivalence between provable and valid, there is no algorithm that, given a first-order formula, could:

- build a proof
- or warn us that this formula has no proof.

# Alonzo Church (1903-1995), american logician

- Inventor of the lambda-calculus (1936)
  $(\lambda x.xy)\,(\lambda z.z) \;\rightarrow_\beta\; (\lambda z.z)y$
  - attempt at a universal computational model
  - basis for functional languages (ML, Lisp...)
  - can represent programs as well as proofs
  - one of the first notions of typing

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- Proof that first-order logic is algorithmically undecidable (hindering strongly Hilbert's program)
- Independently proved by Turing (1937)

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  - attempt at a universal computational model
  - basis for functional languages (ML, Lisp...)
  - can represent programs as well as proofs
  - one of the first notions of typing
- Proof that first-order logic is algorithmically undecidable (hindering strongly Hilbert's program)
- Independently proved by Turing (1937)
- Church-Turing's thesis: the $\lambda$-calculus or the Turing machine express exactly what a mechanical computation is

# Overview

Introduction

Rules and examples

Copy rule

Rules for equality

Conclusion

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Introduction

Rules and examples

Copy rule

Rules for equality

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# Reminder: Propositional rules

## Table 3.1

| Introduction | Elimination |
| --- | --- |
| $\dfrac{\begin{array}{c}[A]\\ \cdots\\ B\end{array}}{A \Rightarrow B} \Rightarrow I$ | $\dfrac{A \quad A \Rightarrow B}{B} \Rightarrow E$ |
| $\dfrac{A \quad B}{A \wedge B} \wedge I$ | $\dfrac{A \wedge B}{A} \wedge E1$ <br> $\dfrac{A \wedge B}{B} \wedge E2$ |
| $\dfrac{A}{A \vee B} \vee I1$ <br> $\dfrac{A}{B \vee A} \vee I2$ | $\dfrac{A \vee B \quad A \Rightarrow C \quad B \Rightarrow C}{C} \vee E$ |
| *Ex falso quodlibet* | |
| $\dfrac{\bot}{A}$ *Efq* | |
| *Reductio ad absurdo* | |
| $\dfrac{\neg\neg A}{A}$ *RAA* | |

# An extension of propositonal natural deduction

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- Still only one rule to remove hypotheses: $\Rightarrow$ I.

## Additional rules about

- quantifiers
- copy
- equality

# Consistency and completeness

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- **Consistency :** $\Gamma \vdash A$ implies $\Gamma \models A$.

  Proved in the next lecture.
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# Consistency and completeness

- **Consistency :** $\Gamma \vdash A$ implies $\Gamma \models A$.

  Proved in the next lecture.
  The main point is to prove that the new rules are consistent.

- **Completeness :** $\Gamma \models A$ implies $\Gamma \vdash A$.
  Assumed without proof.

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# Quantifier rules

An elimination rule and an introduction rule for each quantifier.

- How to use these rules on examples.
- And some mistakes you can make if you don't comply with the use conditions of these rules.

# Reminder

## Definition 4.3.34

Let $x$ be a variable, $t$ a term and $A$ a formula.

1. $A < x := t >$ is the formula obtained by replacing in $A$ every free occurrence of $x$ with the term $t$.
2. The term $t$ is free for $x$ in $A$ if the variables of $t$ are not bound in the free occurrences of $x$.

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## Example

$$A = \forall y P(x, y)$$

- Is $z$ free for $x$ in $A$ ?

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- Is $g(y)$ free for $x$ in $A$ ?

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## Example

$$A = \forall y P(x, y)$$

- Is $z$ free for $x$ in $A$ ? yes
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- Is $f(x)$ free for $y$ in $A$ ?

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## Example

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- Is $g(y)$ free for $x$ in $A$ ? no
- Is $f(x)$ free for $y$ in $A$ ? yes

# Quantifier rules: ∀E

*A* and *B* are formulae, *x* is a variable, *t* is a term

## ∀ Elimination

$$\frac{\forall x A}{A < x := t >}\forall E$$

*t* must be free for *x* in *A*.

# Example 6.1.1

**Wrong use of the rule $\forall$E: where is the mistake ?**

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $\forall x \exists y P(x, y)$ | |
| 1 | 2 | $\exists y P(y, y)$ | $\forall E$ 1, $y$ |
| | 3 | Therefore $\forall x \exists y P(x, y) \Rightarrow \exists y P(y, y)$ | |

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| 1 | 1 | Assume $\forall x \exists y P(x, y)$ | |
| 1 | 2 | $\exists y P(y, y)$ | $\forall E$ 1, $y$ ERROR |
| | 3 | Therefore $\forall x \exists y P(x, y) \Rightarrow \exists y P(y, y)$ | |

On line 2, the use conditions of $\forall$E are not met
because the term $y$ isn't free for $x$ in the formula $\exists y P(x, y)$.

# Example 6.1.1

**Wrong use of the rule ∀E: where is the mistake ?**

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| 1 | 1 | Assume $\forall x \exists y P(x, y)$ | |
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On line 2, the use conditions of ∀E are not met
because the term $y$ isn't free for $x$ in the formula $\exists y P(x, y)$.

Let $I$ be the interpretation with domain $\{0, 1\}$ such that $P_I =$

$\{(0, 1), (1, 0)\}$

This interpretation makes the "conclusion" false.

# Quantifier rules: ∀ I

$A$ and $B$ are formulae, $x$ is a variable.

## ∀ Introduction

$$\frac{A}{\forall x A}\forall I$$

$x$ must be free

- neither in the **environment** of the proof,
- nor in the **context** of the premise of the rule.

# Example 6.1.2 $\forall y P(y) \wedge \forall y Q(y) \Rightarrow \forall x (P(x) \wedge Q(x))$

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| 1 | 1 | Assume $\forall y P(y) \wedge \forall y Q(y)$ | |
| 1 | 2 | $\forall y P(y)$ | $\wedge E1$ 1 |

## Example 6.1.2 $\forall y P(y) \wedge \forall y Q(y) \Rightarrow \forall x (P(x) \wedge Q(x))$

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| 1 | 1 | Assume $\forall y P(y) \wedge \forall y Q(y)$ | |
| 1 | 2 | $\forall y P(y)$ | $\wedge E1$ 1 |
| 1 | 3 | $\forall y Q(y)$ | $\wedge E2$ 1 |

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| | | | |
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| 1 | 1 | Assume $\forall y P(y) \wedge \forall y Q(y)$ | |
| 1 | 2 | $\forall y P(y)$ | $\wedge E1$ 1 |
| 1 | 3 | $\forall y Q(y)$ | $\wedge E2$ 1 |
| 1 | 4 | $P(x)$ | $\forall E$ 2, $x$ |
| 1 | 5 | $Q(x)$ | $\forall E$ 3, $x$ |

**Remark :** When using rule ∀E on lines 4 and 5, we specify that $y$ has been replaced with $x$.

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| 1 | 6 | $P(x) \wedge Q(x)$ | $\wedge I$ 4, 5 |

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| 1 | 2 | $\forall y P(y)$ | $\wedge E1$ 1 |
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| 1 | 6 | $P(x) \wedge Q(x)$ | $\wedge I$ 4, 5 |
| 1 | 7 | $\forall x(P(x) \wedge Q(x))$ | $\forall I$ 6 |

**Remark :** When using rule ∀E on lines 4 and 5, we specify that $y$ has been replaced with $x$.

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| 1 | 5 | $Q(x)$ | $\forall E$ 3, $x$ |
| 1 | 6 | $P(x) \wedge Q(x)$ | $\wedge I$ 4, 5 |
| 1 | 7 | $\forall x(P(x) \wedge Q(x))$ | $\forall I$ 6 |
| | 8 | Therefore $\forall y P(y) \wedge \forall y Q(y) \Rightarrow \forall x(P(x) \wedge Q(x))$ | $\Rightarrow$I 1, 7 |

**Remark :** When using rule ∀E on lines 4 and 5, we specify that $y$ has been replaced with $x$.

# Example 6.1.3

## Wrong use of the rule ∀I

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $P(x)$ | |
| 1 | 2 | $\forall x P(x)$ | $\forall I$ 1 |
| | 3 | Therefore $P(x) \Rightarrow \forall x P(x)$ | $\Rightarrow$I 1, 2 |

# Example 6.1.3

## Wrong use of the rule ∀I

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $P(x)$ | |
| 1 | 2 | $\forall x P(x)$ | $\forall I$ 1 ERROR |
| | 3 | Therefore $P(x) \Rightarrow \forall x P(x)$ | $\Rightarrow$I 1, 2 |

On line 2, $x$ is free in the context $P(x)$, which disallows generalisation on $x$.

# Example 6.1.3

## Wrong use of the rule ∀I

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| 1 | 1 | Assume $P(x)$ | |
| 1 | 2 | $\forall x P(x)$ | $\forall I$ 1 ERROR |
| | 3 | Therefore $P(x) \Rightarrow \forall x P(x)$ | $\Rightarrow$I 1, 2 |

On line 2, $x$ is free in the context $P(x)$, which disallows generalisation on $x$.

Let $I$ be the interpretation with domain $\{0, 1\}$ such that $P_I = \{0\}$.
Let $e$ be a state where $x = 0$.
The assignment $(I, e)$ makes the "conclusion" false.

# Quantifier rules: ∃E

*A* and *B* are formulae, *x* is a variable.

## ∃ Elimination

$$\frac{\exists x A \qquad (A \Rightarrow B)}{B}\exists E$$

*x* must be free

- neither in the environment,
- nor in *B*,
- nor in the context of $A \Rightarrow B$.

# Example 6.1.4

## Wrong use of the rule ∃E

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $\exists x P(x) \wedge (P(x) \Rightarrow \forall y Q(y))$ | |
| 1 | 2 | $\exists x P(x)$ | $\wedge E1$ 1 |
| 1 | 3 | $P(x) \Rightarrow \forall y Q(y)$ | $\wedge E2$ 1 |
| 1 | 4 | $\forall y Q(y)$ | $\exists E$ 2, 3 |
| | 5 | Therefore $\exists x P(x) \wedge (P(x) \Rightarrow \forall y Q(y)) \Rightarrow \forall y Q(y)$ | $\Rightarrow I$ 1,4 |

# Example 6.1.4

## Wrong use of the rule ∃E

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $\exists x P(x) \wedge (P(x) \Rightarrow \forall y Q(y))$ | |
| 1 | 2 | $\exists x P(x)$ | $\wedge E1$ 1 |
| 1 | 3 | $P(x) \Rightarrow \forall y Q(y)$ | $\wedge E2$ 1 |
| 1 | 4 | $\forall y Q(y)$ | $\exists E$ 2, 3 ERROR |
| | 5 | Therefore $\exists x P(x) \wedge (P(x) \Rightarrow \forall y Q(y)) \Rightarrow \forall y Q(y)$ | $\Rightarrow I$ 1,4 |

The context of the premise $P(x) \Rightarrow \forall y Q(y)$ must not depend on $x$.

# Example 6.1.4

## Wrong use of the rule ∃E

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $\exists x P(x) \wedge (P(x) \Rightarrow \forall y Q(y))$ | |
| 1 | 2 | $\exists x P(x)$ | $\wedge E1$ 1 |
| 1 | 3 | $P(x) \Rightarrow \forall y Q(y)$ | $\wedge E2$ 1 |
| 1 | 4 | $\forall y Q(y)$ | $\exists E$ 2, 3 ERROR |
| | 5 | Therefore $\exists x P(x) \wedge (P(x) \Rightarrow \forall y Q(y)) \Rightarrow \forall y Q(y)$ | $\Rightarrow I$ 1,4 |

The context of the premise $P(x) \Rightarrow \forall y Q(y)$ must not depend on $x$.

Let $I$ be the interpretation with domain $\{0, 1\}$ such that $P_I = Q_I = \{0\}$.
Let $e$ be the state where $x = 1$.
The assignment $(I, e)$ makes this “conclusion” false.

# Example 6.1.5

## Wrong use of the rule ∃E

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $\exists x P(x)$ | |
| 1, 2 | 2 | Assume $P(x)$ | |
| 1 | 3 | Therefore $P(x) \Rightarrow P(x)$ | $\Rightarrow I$ 2, 2 |
| 1 | 4 | $P(x)$ | $\exists E$ 1, 3 |
| 1 | 5 | $\forall x P(x)$ | $\forall I$ 4 |
| | 6 | Therefore $\exists x P(x) \Rightarrow \forall x P(x)$ | |

# Example 6.1.5

## Wrong use of the rule ∃E

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $\exists x P(x)$ | |
| 1, 2 | 2 | Assume $P(x)$ | |
| 1 | 3 | Therefore $P(x) \Rightarrow P(x)$ | $\Rightarrow I$ 2, 2 |
| 1 | 4 | $P(x)$ | $\exists E$ 1, 3 ERROR |
| 1 | 5 | $\forall x P(x)$ | $\forall I$ 4 |
| | 6 | Therefore $\exists x P(x) \Rightarrow \forall x P(x)$ | |

The conclusion of rule $\exists E$ must not depend on $x$.

# Example 6.1.5

## Wrong use of the rule ∃E

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $\exists x P(x)$ | |
| 1, 2 | 2 | Assume $P(x)$ | |
| 1 | 3 | Therefore $P(x) \Rightarrow P(x)$ | $\Rightarrow I$ 2, 2 |
| 1 | 4 | $P(x)$ | $\exists E$ 1, 3 ERROR |
| 1 | 5 | $\forall x P(x)$ | $\forall I$ 4 |
| | 6 | Therefore $\exists x P(x) \Rightarrow \forall x P(x)$ | |

The conclusion of rule $\exists E$ must not depend on $x$.

Let $I$ be the interpretation with domain $\{0, 1\}$ such that $P_I = \{0\}$.
$I$ make the "conclusion" false.

# Quantifier rules: ∃I

$A$ and $B$ are formulae, $x$ is a variable, $t$ is a term

## ∃ Introduction

$$\frac{A < x := t >}{\exists x A}\exists I$$

$t$ must be free for $x$ in $A$.

# Example 6.1.6 $\neg\forall xA \Rightarrow \exists x\neg A$ (De Morgan's law)

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| 1 | 1 | Assume $\neg\forall xA$ |
| 1, 2 | 2 | Assume $\neg\exists x\neg A$ |

# Example 6.1.6 $\neg\forall xA \Rightarrow \exists x\neg A$ (De Morgan's law)

| | | |
|---|---|---|
| 1 | 1 | Assume $\neg\forall xA$ |
| 1, 2 | 2 | Assume $\neg\exists x\neg A$ |
| 1, 2, 3 | 3 | Assume $\neg A$ |

# Example 6.1.6 $\neg\forall xA \Rightarrow \exists x\neg A$ (De Morgan's law)

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $\neg\forall xA$ | |
| 1, 2 | 2 | Assume $\neg\exists x\neg A$ | |
| 1, 2, 3 | 3 | Assume $\neg A$ | |
| 1, 2, 3 | 4 | $\exists x\neg A$ | $\exists I$ 3, $x$ |

- On line 4: we use $\neg A = \neg A < x := x >$
  and a variable $x$ is always free for itself in $A$.

# Example 6.1.6 $\neg\forall xA \Rightarrow \exists x\neg A$ (De Morgan's law)

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $\neg\forall xA$ | |
| 1, 2 | 2 | Assume $\neg\exists x\neg A$ | |
| 1, 2, 3 | 3 | Assume $\neg A$ | |
| 1, 2, 3 | 4 | $\exists x\neg A$ | $\exists I$ 3, $x$ |
| 1, 2, 3 | 5 | $\perp$ | $\Rightarrow E$ 2, 4 |

- On line 4: we use $\neg A = \neg A < x := x >$
  and a variable $x$ is always free for itself in $A$.

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|---|---|---|---|
| 1 | 1 | Assume $\neg\forall xA$ | |
| 1, 2 | 2 | Assume $\neg\exists x\neg A$ | |
| 1, 2, 3 | 3 | Assume $\neg A$ | |
| 1, 2, 3 | 4 | $\exists x\neg A$ | $\exists I$ 3, $x$ |
| 1, 2, 3 | 5 | $\perp$ | $\Rightarrow E$ 2, 4 |
| 1, 2 | 6 | Therefore $\neg\neg A$ | $\Rightarrow I$ 3, 5 |

- On line 4: we use $\neg A = \neg A < x := x >$
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| 1, 2, 3 | 3 | Assume $\neg A$ | |
| 1, 2, 3 | 4 | $\exists x\neg A$ | $\exists I$ 3, $x$ |
| 1, 2, 3 | 5 | $\perp$ | $\Rightarrow E$ 2, 4 |
| 1, 2 | 6 | Therefore $\neg\neg A$ | $\Rightarrow I$ 3, 5 |
| 1, 2 | 7 | $A$ | Raa 6 |

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| 1 | 1 | Assume $\neg\forall xA$ | |
| 1, 2 | 2 | Assume $\neg\exists x\neg A$ | |
| 1, 2, 3 | 3 | Assume $\neg A$ | |
| 1, 2, 3 | 4 | $\exists x\neg A$ | $\exists I$ 3, $x$ |
| 1, 2, 3 | 5 | $\perp$ | $\Rightarrow E$ 2, 4 |
| 1, 2 | 6 | Therefore $\neg\neg A$ | $\Rightarrow I$ 3, 5 |
| 1, 2 | 7 | $A$ | Raa 6 |
| 1, 2 | 8 | $\forall xA$ | $\forall I$ 7 |

- On line 4: we use $\neg A = \neg A < x := x >$
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# Example 6.1.6 $\neg\forall xA \Rightarrow \exists x\neg A$ (De Morgan's law)

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| 1 | 1 | Assume $\neg\forall xA$ | |
| 1, 2 | 2 | Assume $\neg\exists x\neg A$ | |
| 1, 2, 3 | 3 | Assume $\neg A$ | |
| 1, 2, 3 | 4 | $\exists x\neg A$ | $\exists I$ 3, $x$ |
| 1, 2, 3 | 5 | $\perp$ | $\Rightarrow E$ 2, 4 |
| 1, 2 | 6 | Therefore $\neg\neg A$ | $\Rightarrow I$ 3, 5 |
| 1, 2 | 7 | $A$ | Raa 6 |
| 1, 2 | 8 | $\forall xA$ | $\forall I$ 7 |
| 1, 2 | 9 | $\perp$ | $\Rightarrow E$ 1, 8 |

- On line 4: we use $\neg A = \neg A < x := x >$
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| 1, 2 | 2 | Assume $\neg\exists x\neg A$ | |
| 1, 2, 3 | 3 | Assume $\neg A$ | |
| 1, 2, 3 | 4 | $\exists x\neg A$ | $\exists I$ 3, $x$ |
| 1, 2, 3 | 5 | $\perp$ | $\Rightarrow E$ 2, 4 |
| 1, 2 | 6 | Therefore $\neg\neg A$ | $\Rightarrow I$ 3, 5 |
| 1, 2 | 7 | $A$ | Raa 6 |
| 1, 2 | 8 | $\forall xA$ | $\forall I$ 7 |
| 1, 2 | 9 | $\perp$ | $\Rightarrow E$ 1, 8 |
| 1 | 10 | Therefore $\neg\neg\exists x\neg A$ | $\Rightarrow$ I 2, 9 |

- On line 4: we use $\neg A = \neg A < x := x >$
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| 1, 2 | 2 | Assume $\neg\exists x\neg A$ | |
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| 1, 2, 3 | 4 | $\exists x\neg A$ | $\exists I$ 3, $x$ |
| 1, 2, 3 | 5 | $\perp$ | $\Rightarrow E$ 2, 4 |
| 1, 2 | 6 | Therefore $\neg\neg A$ | $\Rightarrow I$ 3, 5 |
| 1, 2 | 7 | $A$ | Raa 6 |
| 1, 2 | 8 | $\forall xA$ | $\forall I$ 7 |
| 1, 2 | 9 | $\perp$ | $\Rightarrow E$ 1, 8 |
| 1 | 10 | Therefore $\neg\neg\exists x\neg A$ | $\Rightarrow$ I 2, 9 |
| 1 | 11 | $\exists x\neg A$ | Raa 10 |

- On line 4: we use $\neg A = \neg A < x := x >$
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# Example 6.1.6 $\neg\forall xA \Rightarrow \exists x\neg A$ (De Morgan's law)

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $\neg\forall xA$ | |
| 1, 2 | 2 | Assume $\neg\exists x\neg A$ | |
| 1, 2, 3 | 3 | Assume $\neg A$ | |
| 1, 2, 3 | 4 | $\exists x\neg A$ | $\exists I$ 3, $x$ |
| 1, 2, 3 | 5 | $\bot$ | $\Rightarrow E$ 2, 4 |
| 1, 2 | 6 | Therefore $\neg\neg A$ | $\Rightarrow I$ 3, 5 |
| 1, 2 | 7 | $A$ | Raa 6 |
| 1, 2 | 8 | $\forall xA$ | $\forall I$ 7 |
| 1, 2 | 9 | $\bot$ | $\Rightarrow E$ 1, 8 |
| 1 | 10 | Therefore $\neg\neg\exists x\neg A$ | $\Rightarrow$ I 2, 9 |
| 1 | 11 | $\exists x\neg A$ | Raa 10 |
| | 12 | Therefore $\neg\forall xA \Rightarrow \exists x\neg A$ | $\Rightarrow$ I 1, 11 |

- On line 4: we use $\neg A = \neg A < x := x >$
  and a variable $x$ is always free for itself in $A$.

# Quantifier rules recap: Figure 6.1

| | | |
|---|---|---|
| $\dfrac{A}{\forall x A}$ | $\forall I$ | $x$ must be free<br>- neither in the environment of the proof,<br>- nor in the context of the premise |
| $\dfrac{\forall x A}{A{<}x{:=}t{>}}$ | $\forall E$ | $t$ must be free for $x$ in $A$ |
| $\dfrac{A{<}x{:=}t{>}}{\exists x A}$ | $\exists I$ | $t$ must be free for $x$ in $A$ |
| $\dfrac{\exists x A \qquad (A \Rightarrow B)}{B}$ | $\exists E$ | $x$ must be free<br>- neither in the environment<br>- nor in $B$,<br>- nor in the context of $A \Rightarrow B$ |

# Overview

Introduction

Rules and examples

Copy rule

Rules for equality

Conclusion

# Definition

The copy rule consists in deducing, from a given formula, another formula which is equal up to renaming bound variables.

$$\frac{A'}{A} \text{ copy}$$

# Reminders : Renaming of bound variables (1/3)

Two formulae are $\alpha$-equivalent if one can be transformed into the other by replacing subformulae such as $Qx\ A$ with $Qy\ A < x := y >$ where $Q$ is a quantifier and $y$ does not appear in $Qx\ A$.

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## Example 4.4.4

- $\forall x\ p(x, z) =_\alpha \forall y\ p(y, z)$.
- $\forall x\ p(x, z) \neq_\alpha \forall z\ p(z, z)$.

# Renaming of bound variables (2/3)

**Definition 4.4.5**

Two formulae are equal up to renaming of bound variables if we can obtain one starting from the other by replacements such as 1

$Qx\ A \equiv Qy\ A < x := y >$ where $y$ is a variable not appearing in $Qx\ A$

The two formulae are said to be:

- $\alpha$-equivalent
- or a copy of each other
- denoted $A =_\alpha B$

# Renaming of bound variables (3/3)

**Theorem 4.4.6**

If two formulae are equal up to renaming of bound variables then they are equivalent.

**Example 4.4.7**

Let us show that $\forall x \exists y P(x,y)$ and $\forall y \exists x P(y,x)$ are equivalent.

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$$\begin{array}{rl} & \forall x \exists y P(x, y) \\ =_\alpha & \forall u \exists y P(u, y) \end{array}$$

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$$
\begin{aligned}
& \forall x \exists y P(x, y) \\
=_\alpha \; & \forall u \exists y P(u, y) \\
=_\alpha \; & \forall u \exists x P(u, x) \\
=_\alpha \; & \forall y \exists x P(y, x)
\end{aligned}
$$

# $\alpha$-equivalence howto

## Technique

- Draw lines between each quantifier and the variables that it binds.
- Erase the name of bound variables.

If after this transformation, the two formulae become identical, then they are $\alpha$-equivalent.

## Example 4.4.8

With the two formulae $\forall x \exists y P(y,x)$ and $\forall y \exists x P(x,y)$ :

$$\forall \ x \ \exists \ y \ P( \ y \ , \ x \ )$$

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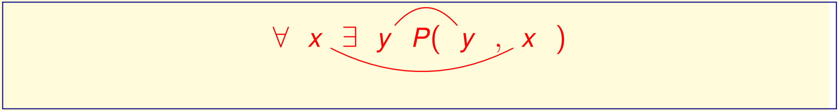

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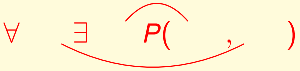

# Exercise

**Compute the transformation for**

- $A = \forall x \forall y\ R(x, y, y)$
- $B = \forall x \forall y\ R(x, x, y)$

Are $A$ and $B$ $\alpha$-equivalent?

# Proof without the copy rule

In the environment $(i)\ \exists x P(x)$ :

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| | 3 | Therefore $P(x) \Rightarrow \exists y P(y)$ | $\Rightarrow$I 1, 2 |

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| | 4 | $\exists y P(y)$ | $\exists$E i, 3 |

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**Theorem (assumed)**

Let $A$ and $A'$ be two formulae which are copies of one another.
Then there exists a proof of $A$ in the environment $A'$.

The copy rule is a derivable rule: its use can always be replaced by a (possibly long) proof.

It is the only derivable rule we will allow.

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# Reflexivity and congruence

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- every term is equal to itself
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| | | |
|---|---|---|
| $\dfrac{}{t{=}t}$ | reflexivity | $t$ is a term |
| $\dfrac{s{=}t \qquad A{<}x{:=}s{>}}{A{<}x{:=}t{>}}$ | congruence | $s$ and $t$ are two terms free for the variable $x$ in the formula $A$ |

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Let us prove that $s = t \Rightarrow t = s$ (symmetry)

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| 1 | 1 | Assume $s = t$ |

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| | | | |
|---|---|---|---|
| 1 | 1 | Assume $s = t$ | |
| 1 | 2 | $s = s$ | reflexivity |

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| | | | |
|---|---|---|---|
| 1 | 1 | Assume $s = t$ | |
| 1 | 2 | $s = s$ | reflexivity |
| 1 | 3 | $t = s$ | congruence 1, 2 $\dfrac{s = t \quad (x = s) < x := s >}{(x = s) < x := t >}$ |

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| | 3 | Therefore $s = t \Rightarrow t = s$ | $\Rightarrow$I 1, 3 |

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| 1 | 2 | $s = s$ | reflexivity |
| 1 | 3 | $t = s$ | congruence 1, 2 $\dfrac{s = t \quad (x = s) < x := s >}{(x = s) < x := t >}$ |
| | 3 | Therefore $s = t \Rightarrow t = s$ | $\Rightarrow$I 1, 3 |

**Remark :** The variable $x$ does not appear in the proof, its only use is to name the place where we replace $s$ with $t$.

In the next examples, we will just underline this place.

# Example 6.1.8

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| | | | |
|---|---|---|---|
| 1 | 1 | Assume $s = t \wedge t = u$ | |
| 1 | 2 | $s = t$ | $\wedge$E1 1 |

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Let us prove that $s = t \wedge t = u \Rightarrow s = u$ (transitivity)

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $s = t \wedge t = u$ | |
| 1 | 2 | $s = t$ | $\wedge$E1 1 |
| 1 | 3 | $t = u$ | $\wedge$E2 1 |

# Example 6.1.8

**Let us prove that $s = t \wedge t = u \Rightarrow s = u$ (transitivity)**

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $s = t \wedge t = u$ | |
| 1 | 2 | $s = \underline{t}$ | $\wedge$E1 1 |
| 1 | 3 | $t = u$ | $\wedge$E2 1 |
| 1 | 4 | $s = \underline{u}$ | congruence 3, 2 |

# Example 6.1.8

**Let us prove that $s = t \wedge t = u \Rightarrow s = u$ (transitivity)**

| | | | |
|---|---|---|---|
| 1 | 1 | Assume $s = t \wedge t = u$ | |
| 1 | 2 | $s = \underline{t}$ | $\wedge$E1 1 |
| 1 | 3 | $t = u$ | $\wedge$E2 1 |
| 1 | 4 | $s = \underline{u}$ | congruence 3, 2 |
| | 5 | Therefore $s = t \wedge t = u \Rightarrow s = u$ | $\Rightarrow$I 1, 4 |

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Rules and examples

Copy rule

Rules for equality

Conclusion

# Today

- First-order resolution is **complete**, and one way to build a first-order proof is by lifting a propositional proof.
- First-order Natural Deduction
  - New rules for introducing and eliminating the quantifiers.
  - Copy, equality

# Next lecture

- Tactics
- Consistency of the system