Propositional Resolution

Second Part: Algorithms

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Proof by resolution of our running example

- (H1) : $p \Rightarrow \neg j \;\equiv\; \neg p \vee \neg j$
- (H2) : $\neg p \Rightarrow j \;\equiv\; p \vee j$
- (H3) : $j \Rightarrow m \;\equiv\; \neg j \vee m$
- (¬ C): $\neg m \wedge \neg p$

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Clauses: $\{\neg p \vee \neg j,\ p \vee j,\ \neg j \vee m,\ \neg m,\ \neg p\}$

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Clauses: $\{\neg p \vee \neg j,\ p \vee j,\ \neg j \vee m,\ \neg m,\ \neg p\}$

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OR...

Last course

- Boolean Algebra
- Boolean functions
- Resolution

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B is a *consequence* of A: every model of A is also a model of B.

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Today: Completeness

(2) $\Rightarrow$ (1)

Overview

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Correctness

Completeness

Introduction to resolution algorithms

The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Complete strategy

Conclusion

Definition

The correctness of a deductive system states that all proofs obtained in this system “prove only true statements”.

Correctness of the resolution rule

Theorem 2.1.15

If C is a resolvent of A and B then $A, B \models C$.

Proof.

□

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If C is a resolvent of A and B, then there is a literal L such that $L \in A, L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.

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Let v be an assignment such that $[A]_v = 1$ and $[B]_v = 1$: let us show that $[C]_v = 1$.

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- Suppose that $[L]_v = 1$.
- Suppose that $[L^c]_v = 1$.

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Let v be an assignment such that $[A]_v = 1$ and $[B]_v = 1$: let us show that $[C]_v = 1$.

- Suppose that $[L]_v = 1$. Therefore $[L^c]_v = 0$.
  Since $[B]_v = 1$, v is a model of a literal of $(B - \{L^c\})$. Hence $[C]_v = 1$.
- Suppose that $[L^c]_v = 1$.

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- Suppose that $[L]_v = 1$. Therefore $[L^c]_v = 0$.
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- Suppose that $[L^c]_v = 1$. Therefore $[L]_v = 0$.
  Since $[A]_v = 1$, v is a model of $(A - \{L\})$. Hence $[C]_v = 1$.

Since every truth assignment is either model of L or L^c, v is a model of C.

☐

Correctness of deduction

Theorem 2.1.16

Let Γ be a set of clauses and C a clause. If $\Gamma \vdash C$ then $\Gamma \models C$.

Proof.

Suppose that there is a proof P of C starting from Γ.
Suppose that for any proof of $\Gamma \vdash D$ shorter than P, we have $\Gamma \models D$.
Let us show that $\Gamma \models C$. There are two possible cases:

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1. C is a member of Γ, in this case $\Gamma \models C$.

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Suppose that for any proof of $\Gamma \vdash D$ shorter than P, we have $\Gamma \models D$.
Let us show that $\Gamma \models C$. There are two possible cases:

1. C is a member of Γ, in this case $\Gamma \models C$.
2. $\Gamma \vdash A$ and $\Gamma \vdash B$ (with a shorter proof) and

$$\frac{A \qquad B}{C}$$

By induction hypothesis: $\Gamma \models A$ and $\Gamma \models B$.

By correctness of the resolution rule: $A, B \models C$. Hence $\Gamma \models C$.

☐

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Completeness for the refutation is the property: If $\Gamma \models \bot$ then $\Gamma \vdash \bot$.

We prove this result for a finite Γ.

$\Gamma[L := 1]$

Definition 2.1.18

Let Γ be a set of clauses and L a literal.

$\Gamma[L := 1]$ is obtained by:

- deleting the clauses containing L
- removing L^c from the other clauses.

$\Gamma[L := 0]$ is similarly defined by switching the roles of L and L^c.

Remark: the number of variables in Γ has been decreased.

Examples

Example 2.1.19

Let Γ be the set of clauses $\overline{p}+q$, $\overline{q}+r$, $p+q$, $p+r$. We have:

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- $\Gamma[p:=0] =$

  $\boxed{\{\overline{q}+r,q,r\}.}$

Notice that:

- $(\overline{1}+q)(\overline{q}+r)(1+q)(1+r) \equiv$

  $\boxed{q(\overline{q}+r)}$

- $(\overline{0}+q)(\overline{q}+r)(0+q)(0+r) \equiv$

  $\boxed{(\overline{q}+r)qr}$

Property of $\Gamma[L := ...]$

Property 2.1.21

Γ has a model if and only if $\Gamma[L := 1]$ or $\Gamma[L := 0]$ has a model.

Proof.

☐

Property of $\Gamma[L := ...]$

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Γ has a model if and only if $\Gamma[L := 1]$ or $\Gamma[L := 0]$ has a model.

Proof.

$\Rightarrow$ If v is a model of Γ then v is a model of
either $\Gamma[L := 0]$ (if $[L]_{v'} = 0$)
or $\Gamma[L := 1]$ (if $[L]_{v'} = 1$)

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Property of $\Gamma[L := \ldots]$

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Γ has a model if and only if $\Gamma[L := 1]$ or $\Gamma[L := 0]$ has a model.

Proof.

$\Rightarrow$ If v is a model of Γ then v is a model of
either $\Gamma[L := 0]$ (if $[L]_{v'} = 0$)
or $\Gamma[L := 1]$ (if $[L]_{v'} = 1$)

$\Leftarrow$ If v is a model of $\Gamma[L := i]$
then we can build a model of Γ (by taking $[L]_{v'} = i$)

□

Lemma 2.1.22

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Let Γ a set of clauses, C a clause and L a literal.
If $\Gamma[L := 1] \vdash C$ then $\Gamma \vdash C$ or $\Gamma \vdash C + L^c$.

Proof.

Idea: we put back L^c in the clauses where it was removed.

- If $C \in \Gamma[L := 1]$:
- If C is a resolvent of A and B:

□

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- If $C \in \Gamma[L := 1]$:
  - either C was in Γ, thus $\Gamma \vdash C$
  - or C was obtained by removing a L^c, thus $\Gamma \vdash C + L^c$
- If C is a resolvent of A and B:

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Let Γ a set of clauses, C a clause and L a literal.
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Proof.

Idea: we put back L^c in the clauses where it was removed.

- If $C \in \Gamma[L := 1]$:
  - either C was in Γ, thus $\Gamma \vdash C$
  - or C was obtained by removing a L^c, thus $\Gamma \vdash C + L^c$
- If C is a resolvent of A and B:
  - either $\Gamma \vdash A$ and $\Gamma \vdash B$, hence $\Gamma \vdash C$
  - or L^c has to be put back into A or B, thus into C too

□

Completeness of propositional resolution

Theorem 2.1.24

Let Γ be a finite set of clauses. If Γ is unsatisfiable then $\Gamma \vdash \bot$.

Proof

By induction on the number of variables in Γ.

- Base case: Γ has no variable, so $\Gamma = \emptyset$ (impossible, it's valid) or $\Gamma = \{\bot\}$.
- Inductive step: either we prove directly that $\Gamma \vdash \bot$, or that $\Gamma \vdash x$ and $\Gamma \vdash \overline{x}$.

Corollary 2.1.25

Γ is unsatisfiable if and only if $\Gamma \vdash \bot$.

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Presentation of two algorithms

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Presentation of two algorithms

How to "systematically" decide whether Γ is inconsistent or not?

- The Davis-Putnam-Logemann-Loveland Algorithm
  "Intelligent" traversal of the possible assignments of Γ
- Complete strategy
  Construction of ALL the deductible clauses (resolvents) from Γ

Remark

Exponential solutions in time in the worst case.

Exponential complexity

Remember that two clauses having the same set of literals are equal.

If Γ uses n, then we have at most 2^n distinct clauses deduced from Γ.

Reduction of a set of clauses

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Example 2.1.27

The reduction of the set of clauses $\{p+q+\overline{p}, p+r, p+r+\overline{s}, r+q\}$ gives the reduced set:

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The reduction of the set of clauses $\{p+q+\overline{p}, p+r, p+r+\overline{s}, r+q\}$ gives the reduced set:

$$\boxed{\{\cancel{p+q+\overline{p}}, p+r, \cancel{p+r+\overline{s}}, r+q\}.}$$

Justification

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A set of clauses E is equivalent to the reduced set of clauses obtained from E.

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A set of clauses E is equivalent to the reduced set of clauses obtained from E.

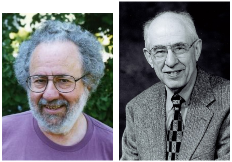
Proof.

- Removing valid clauses: $x.1 \equiv x$
- Removing a clause including another clause: $x(x+y) \equiv x$

☐

Overview

History



- Martin Davis (1928-), american mathematician
- Hilary Putnam (1926-2016), american philosopher, mathematician and computer scientist
- **resolution** rule (exhaustively used in the first algorithm)
- Algorithm for satisfiability of boolean formulas (1960)
  - finds (if possible) a model of a set of clauses
  - initially devised to study first-order formulas
  - refined in 1962 by M. Davis, G. Logemann and D. Loveland with a branching mechanism
  - Basis for efficient SAT-solvers
- Proof of undecidability of Diophantine equations (with Y. Matiyasevich and J. Robinson)

Principle I

Two types of formulae transformations:

1. preserving the truth value:
    - reduction

Principle I

Two types of formulae transformations:

1. preserving the truth value:
    - reduction
2. preserving only satisfiability:
    - pure literal elimination
    - unit resolution

DPLL is (usually) efficient because it uses these two kinds transformations.

Principle II

“Branching/Backtracking” (splitting rule)

- **Branching**: After simplification, assign to **true** a heuristically chosen variable (branching literal).
- Continue the algorithm recursively.

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“Branching/Backtracking” (splitting rule)

- **Branching**: After simplification, assign to **true** a heuristically chosen variable (branching literal).
- Continue the algorithm recursively.
- **Backtracking**: If we arrive to a contradiction, we return to the last choice, and we “branch” by assigning **false** to the chosen variable.

The DPLL Algorithm (figure 2.1)

```
bool function Algo_DPLL( $\Gamma$: set of clauses)
0   Remove the valid clauses from $\Gamma$.
    If $\Gamma = \emptyset$, return (true).
    Else return (DPLL($\Gamma$))

bool function DPLL( $\Gamma$: set of non-valid clauses)
The function returns true if and only if $\Gamma$ is satisfiable.
1   If $\bot \in \Gamma$, return(false).
    If $\Gamma = \emptyset$, return (true).
2   Reduce $\Gamma$.
3   Remove from $\Gamma$ the clauses containing a pure literal.
    If the set $\Gamma$ has been modified, goto 1.
4   Apply unit resolution to $\Gamma$.
    If the set $\Gamma$ has been modified, goto 1.
5   Pick an arbitrary variable $x$ in $\Gamma$
    return (DPLL($\Gamma[x := 0]$) or else DPLL($\Gamma[x := 1]$))
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The function returns true if and only if Γ is satisfiable.
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2   Reduce Γ.
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Removal of clauses containing a pure literal

Definition 2.3.1

A litteral L is **pure** if none of the clauses in Γ contains L^c.

Lemma 2.3.2

Removing clauses with a pure literal preserves satisfiability.

Proof: see exercise 49.

Intuition: assigning $[L]_v$ to 1 is always possible for a pure literal.

Example 2.3.3

Let Γ be the set of clauses

(1) $p+q+r$

(2) $\overline{q}+\overline{r}$

(3) $q+s$

(4) $\overline{s}+t$

Simplify Γ by removing clauses containing pure literals.

Example 2.3.3

Let Γ be the set of clauses

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Simplify Γ by removing clauses containing pure literals.

The literals p and t are pure.
Therefore we obtain

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Example 2.3.3

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The literals $\overline{r}$ and s are pure.

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The literals p and t are pure.
Therefore we obtain

(2) $\overline{q} + \overline{r}$

(3) $q + s$

The literals $\overline{r}$ and s are pure.
We obtain the empty set.

Example 2.3.3

Let Γ be the set of clauses

(1) $p + q + r$

(2) $\overline{q} + \overline{r}$

(3) $q + s$

(4) $\overline{s} + t$

Simplify Γ by removing clauses containing pure literals.

The literals p and t are pure.
Therefore we obtain

(2) $\overline{q} + \overline{r}$

(3) $q + s$

The literals $\overline{r}$ and s are pure.
We obtain the empty set.
Therefore Γ has a model (for instance $p = 1, t = 1, r = 0, s = 1$).

Unit resolution

Definition 2.3.4

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Lemma 2.3.5

Let L be the literal from a unit clause of Γ.
Let Θ be the set of clauses obtained by:

- removing the clauses containing L
- removing L^c **inside** the remaining clauses

- if Γ contains two complementary unit clauses, then $\Theta = \{\bot\}$.

We apply this process for every unit clause.
Γ has a model if and only if Θ has a model.

Proof: The proof is requested in exercise 50.

Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

- $\Gamma = p + q,\ \bar{p},\ \bar{q}$

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  $q,\ \bar{q}$ by unit resolution on $\bar{p}$, then $\perp$ by UR on $\bar{q}$
  Hence Γ

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- $\Gamma = a + b + \bar{d},\ \bar{a} + c + \bar{d},\ \bar{b},\ d,\ \bar{c}$

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- $\Gamma = a + b + \bar{d},\ \bar{a} + c + \bar{d},\ \bar{b},\ d,\ \bar{c}$

  1. $a,\ \bar{a}$.

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Simplify the following sets of clauses by unit resolution:

- $\Gamma = p+q,\ \bar{p},\ \bar{q}$

  $q,\ \bar{q}$ by unit resolution on $\bar{p}$, then $\perp$ by UR on $\bar{q}$
  Hence Γ has no model.

- $\Gamma = a+b+\bar{d},\ \bar{a}+c+\bar{d},\ \bar{b},\ d,\ \bar{c}$

  1. $a, \bar{a}$.
  2. $\perp$

  hence Γ has no model.

- $\Gamma = p,\ q,\ p+r,\ \bar{p}+r,\ q+\bar{r},\ \bar{q}+s$

Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

- $\Gamma = p + q,\ \bar{p},\ \bar{q}$

  $q,\ \bar{q}$ by unit resolution on $\bar{p}$, then $\bot$ by UR on $\bar{q}$
  Hence Γ has no model.

- $\Gamma = a + b + \bar{d},\ \bar{a} + c + \bar{d},\ \bar{b},\ d,\ \bar{c}$

  1. $a, \bar{a}$.
  2. $\bot$

  hence Γ has no model.

- $\Gamma = p,\ q,\ p + r,\ \bar{p} + r,\ q + \bar{r},\ \bar{q} + s$

  By unit resolution, we obtain: r, s.

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  hence Γ has no model.

- $\Gamma = p,\ q,\ p + r,\ \bar{p} + r,\ q + \bar{r},\ \bar{q} + s$

  By unit resolution, we obtain: r, s.
  This set of clauses has a model, hence Γ has a model.

Removal of valid clauses

Lemma 2.3.7

Let Θ be the set of clauses obtained by removing the valid clauses of Γ.

$$\Gamma \text{ has a model iff } \Theta \text{ has a model.}$$

Proof.

$\Rightarrow$ Every model of Γ is clearly a model of Θ, since $\Theta \subseteq \Gamma$.

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Proof.

$\Rightarrow$ Every model of Γ is clearly a model of Θ, since $\Theta \subseteq \Gamma$.

$\Leftarrow$ Suppose that Θ has a model v.

Let v' be the truth assignment built from v by assigning any value to the variables appearing in Γ but not in Θ.

Every clause C in Γ is:

- either a clause of Θ, then $[C]'_v = [C]_v = 1$
- or a valid clause, so obviously v' is a model of C.

Hence v' is a model of Γ.

☐

The DPLL Algorithm (figure 2.1)

```
bool function Algo_DPLL( $\Gamma$: set of clauses)
0   Remove the valid clauses from $\Gamma$.
    If $\Gamma = \emptyset$, return (true).
    Else return (DPLL($\Gamma$))

bool function DPLL( $\Gamma$: set of non-valid clauses)
The function returns true if and only if $\Gamma$ is satisfiable.
1   If $\bot \in \Gamma$, return(false).
    If $\Gamma = \emptyset$, return (true).
2   Reduce $\Gamma$.
3   Remove from $\Gamma$ the clauses containing a pure literal.
    If the set $\Gamma$ has been modified, goto 1.
4   Apply unit resolution to $\Gamma$.
    If the set $\Gamma$ has been modified, goto 1.
5   Pick an arbitrary variable $x$ in $\Gamma$
    return (DPLL($\Gamma[x := 0]$) or else DPLL($\Gamma[x := 1]$))
```

Example 2.3.8

Let Γ be the set of clauses: $\overline{a}+\overline{b}$, $a+b$, $\overline{a}+\overline{c}$, $a+c$, $\overline{b}+\overline{c}$, $b+c$.

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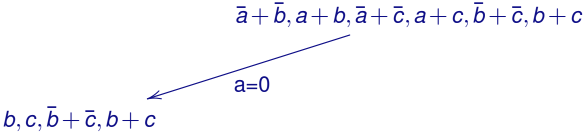

$\bar{a}+\bar{b}, a+b, \bar{a}+\bar{c}, a+c, \bar{b}+\bar{c}, b+c$

a=0

$b, c, \bar{b}+\bar{c}, b+c$

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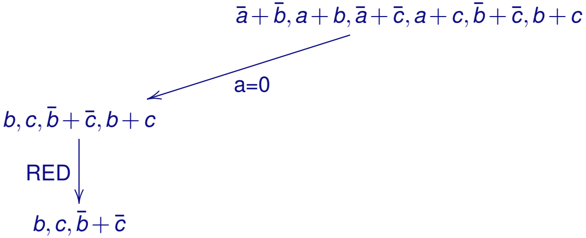

$\bar{a}+\bar{b}, a+b, \bar{a}+\bar{c}, a+c, \bar{b}+\bar{c}, b+c$

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$b, c, \bar{b}+\bar{c}, b+c$

RED

$b, c, \bar{b}+\bar{c}$

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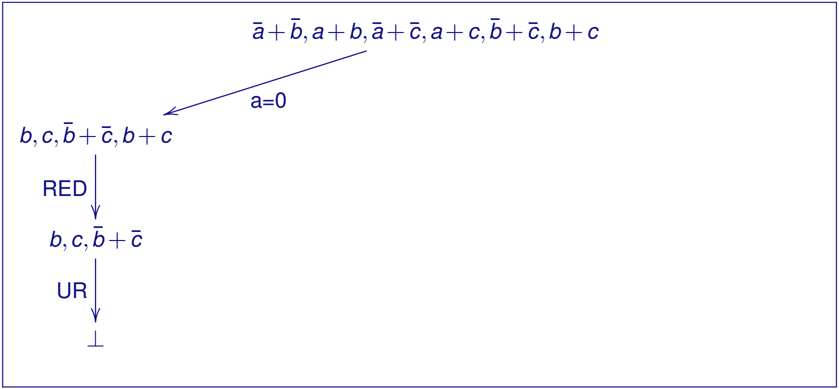

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$\perp$

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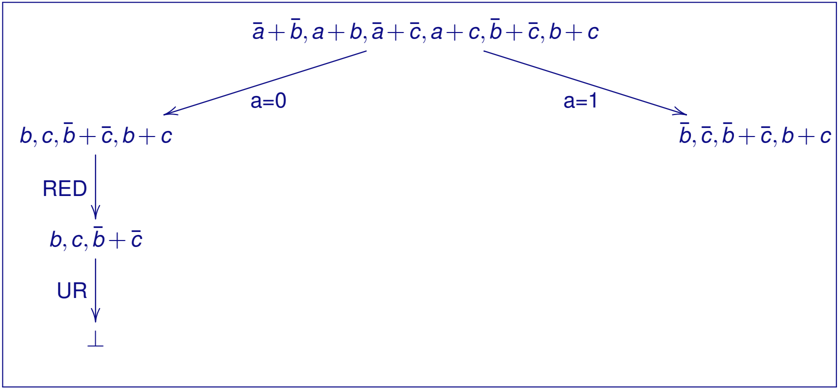

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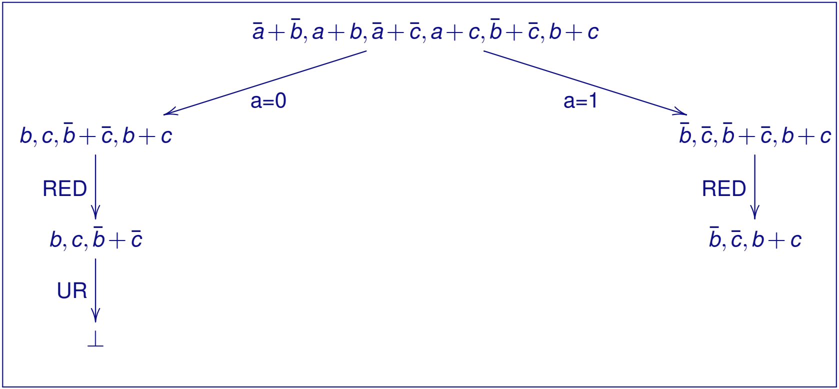

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$\bot$

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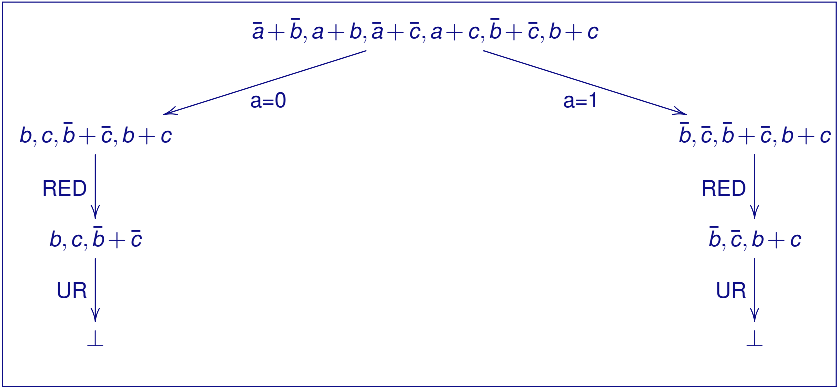

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$b, c, \bar{b}+\bar{c}, b+c$

$\bar{b}, \bar{c}, \bar{b}+\bar{c}, b+c$

RED

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$\bar{b}, \bar{c}, b+c$

UR

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$\perp$

$\perp$

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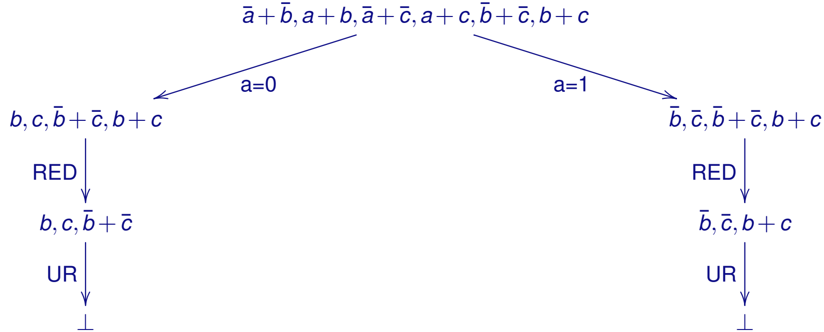

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$\bot$

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Since every leave contains the empty clause, the set Γ is unsatisfiable.

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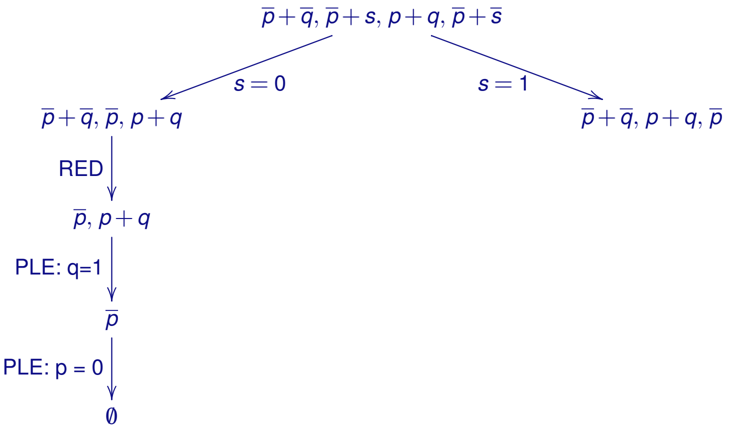

$\overline{p}+\overline{q}$, $\overline{p}+s$, $p+q$, $\overline{p}+\overline{s}$

$s=0$

$\overline{p}+\overline{q}$, $\overline{p}$, $p+q$

RED

$\overline{p}$, $p+q$

PLE: q=1

$\overline{p}$

PLE: p = 0

$\emptyset$

$s=1$

$\overline{p}+\overline{q}$, $p+q$, $\overline{p}$

Since one branch leads to the empty set, the set Γ is satisfiable.

It is useless to continue the construction of the right branch.

Theorems 2.3.9 et 2.3.10

The algorithm Algo_DPLL is correct and terminates.

Theorems 2.3.9 et 2.3.10

The algorithm Algo_DPLL is correct and terminates.

Termination proof

- Valid clause removal is only executed once
- Simplification iteration: the number of clauses strictly decreases
- Recursive calls: the number of variables strictly decreases

Hence the termination.

Correctness proof

- Invariant for the simplification loop:
  $\boxed{\text{the current value of } \Gamma \text{ has a model iff } \Gamma \text{ has a model.}}$

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  Reminder of property 2.1.21:
  Γ *has a model iff* $\Gamma[x := 0]$ *or* $\Gamma[x := 1]$ *is satisfiable.*
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  Reminder of property 2.1.21:
  Γ *has a model iff* $\Gamma[x := 0]$ *or* $\Gamma[x := 1]$ *is satisfiable.*
  So if the recursive calls are correct, the current call is too.

Since the algorithm is correct for a set Γ with no literal, it is correct for any set Γ of clauses.

Remarks 2.3.11 and 2.3.12

- **Forgetting simplifications:** DPLL is still correct if we forget (once or more) reduction (2), pure literal elimination (3) and/or unit reduction (4).

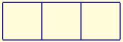
Remarks 2.3.11 and 2.3.12

- **Forgetting simplifications:** DPLL is still correct if we forget (once or more) reduction (2), pure literal elimination (3) and/or unit reduction (4).
- **Choice of the variable (branching literal):**
    - A good choice for variable x in step (5) is the variable that appears most often.
    - A better choice is the variable which will lead to the maximum number of simplifications

Cf. Sub-section 2.3.5, for the main branching heuristics

SAT Solveur demo

Problem



- Each square may either contain a token or not.
- Two neighbouring squares can never both contain a token.
- At least two squares must contain a token.

SAT Solveur demo

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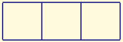

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Input of the problem: the length n of the grid

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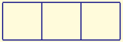

- Each square may either contain a token or not.
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Input of the problem: the length n of the grid

Boolean modelization

- Each square is associated to a boolean variable (true if the square contains a token)
- For the Dimacs format, we number the squares 1 to n

Overview

Principle of the algorithm: Build all the clauses deduced from Γ

Following the height of the proof trees.

Algorithm

For any integer i
While it is possible to construct new clauses
Build the reduced set of all the clauses having a proof tree of height at most i.

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Algorithm

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While it is possible to construct new clauses
Build the reduced set of all the clauses having a proof tree of height at most i.

In practice:
Maintain two sequences of the sets of clauses, $\Delta_{i(i \geq 0)}$ and $\Theta_{i(i \geq 0)}$

Result of the algorithm: minimum deduction clauses

Definition 2.1.29

A minimum clause for the deduction from Γ is :

- a non-valid clause
- deduced from Γ
- and containing no other clause deduced from Γ.

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$\Gamma = \{a+\overline{b}, b+c+d\}$

- The clause $a+c+d$ is a minimum clause for deduction.

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- deduced from Γ
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Example 2.1.30

$\Gamma = \{a + \overline{b}, b + c + d\}$

- The clause $a + c + d$ is a minimum clause for deduction.
- But if we add $\overline{a} + c$ to Γ, then $a + c + d$ is not minimal anymore (since we can now deduce $c + d$).

Property

Property 2.1.31

Let Θ be the set of minimum deduction clauses for the set Γ.
Γ is unsatisfiable if and only if $\perp \in \Theta$.

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- Suppose $\perp \in \Theta$, then $\Gamma \vdash \perp$, hence by resolution correctness, Γ is unsatisfiable.

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Let Θ be the set of minimum deduction clauses for the set Γ.
Γ is unsatisfiable if and only if $\bot \in \Theta$.

Proof.

- Suppose $\bot \in \Theta$, then $\Gamma \vdash \bot$, hence by resolution correctness, Γ is unsatisfiable.
- Suppose Γ is unsatisfiable, by resolution completeness, $\Gamma \vdash \bot$. Consequently $\bot$ is a minimum clause for deduction from Γ, therefore $\bot \in \Theta$.

□

Two sequences of sets of clauses

Δ_i are the new useful clauses

Clauses deduced from Γ by a proof of height i, after removal of:

- valid clauses
- clauses including another clause whose proof has height $< i$.

Δ_0 is obtained by reducing Γ.

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Θ_i are the old clauses still useful

Clauses deduced from Γ by a proof of height $< i$ after removal of:

- valid clauses
- clauses including another clause whose proof has height $\leq i$.

Θ_0 is the empty set.

Construction of the sequences $\Delta_{i(i\geq 0)}$ and $\Theta_{i(i\geq 0)}$

Δ_{i+1}

- Compute all the resolvents of Δ_i and $\Delta_i \cup \Theta_i$
- Reduce this set
- Remove the new resolvents including a clause from $\Delta_i \cup \Theta_i$

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Θ_{i+1}

Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause from Δ_{i+1}.

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Θ_{i+1}

Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause from Δ_{i+1}.

When $\Delta_k = \emptyset$, stop the construction:

- $k - 1$ is then the maximum height of a proof
- Θ_k is the reduced set of the clauses deduced from Γ

Exemple 2.2.1

Soit $\Gamma = \{a+b+\bar{a}, a+b, a+b+c, a+\bar{b}, \bar{a}+b, \bar{a}+\bar{b}\}$

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of Δ_i and $\Delta_i \cup \Theta_i$
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  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$
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  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of Δ_{i+1}.

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Soit $\Gamma = \{a+b+\bar{a}, a+b, a+b+c, a+\bar{b}, \bar{a}+b, \bar{a}+\bar{b}\}$

i	Δ_i	Θ_i	$\Delta_i \cup \Theta_i$	Résolvants de Δ_i et $\Delta_i \cup \Theta_i$

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i	Δ_i	Θ_i	$\Delta_i \cup \Theta_i$	Résolvants de Δ_i et $\Delta_i \cup \Theta_i$
0				

Rappel :

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0	$\cancel{a+b+\bar{a}}, a+b,$ $\cancel{a+b+c}, a+\bar{b}$ $\bar{a}+b, \bar{a}+\bar{b}$	$\emptyset$	$a+b, a+\bar{b},$ $\bar{a}+b, \bar{a}+\bar{b}$	

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0	~~$a+b+\bar{a}$~~, $a+b$ ~~$a+b+c$~~, $a+\bar{b}$ $\bar{a}+b, \bar{a}+\bar{b}$	$\emptyset$	$a+b, a+\bar{b},$ $\bar{a}+b, \bar{a}+\bar{b}$	$a, b,$ ~~$b+\bar{b}$~~, ~~$a+\bar{a}$~~, $\bar{b}, \bar{a}$
1				

Rappel :

- $\Delta_{i+1} =$
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Exemple 2.2.1

Soit $\Gamma = \{a+b+\bar{a}, a+b, a+b+c, a+\bar{b}, \bar{a}+b, \bar{a}+\bar{b}\}$

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0	~~$a+b+\bar{a}$~~, $a+b$ ~~$a+b+c$~~, $a+\bar{b}$ $\bar{a}+b, \bar{a}+\bar{b}$	$\emptyset$	$a+b, a+\bar{b}$, $\bar{a}+b, \bar{a}+\bar{b}$	a, b, ~~$b+\bar{b}$~~, ~~$a+\bar{a}$~~, $\bar{b}, \bar{a}$
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The proof we built

1	$a+b$	
2	$a+\overline{b}$	
3	$\overline{a}+b$	
4	$\overline{a}+\overline{b}$	
5	a	resolvent of 1 and 2
6	b	resolvent of 1 and 3
7	$\overline{b}$	resolvent of 2 and 4
8	$\overline{a}$	resolvent of 3 and 4
9	$\perp$	resolvent of 5 and 8

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Example 2.2.2

$$\{a, c, \overline{a}+\overline{b}, \overline{c}+e\}$$

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i	Δ_i	Θ_i	$\Delta_i \cup \Theta_i$	Rés. de Δ_i et $\Delta_i \cup \Theta_i$
0	$a, c, \overline{a}+\overline{b}, \overline{c}+e$	$\emptyset$	$a, c, \overline{a}+\overline{b}, \overline{c}+e$	$\overline{b}, e$

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1	$\overline{b}, e$	a, c	$\overline{b}, e, a, c$	$\emptyset$

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1	$\overline{b}, e$	a, c	$\overline{b}, e, a, c$	$\emptyset$
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Termination of the algorithm: idea

There are at most 2^n clauses deduced from Γ.

$\Delta_{i(i \geq 0)}$ contains only clauses deduced from Γ

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For all $i \leq k$, the sets Δ_i are mutually disjoint.
(by construction of Δ_i)

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There are at most 2^n clauses deduced from Γ.

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Property 2.2.4

For all $i \leq k$, the sets Δ_i are mutually disjoint.
(by construction of Δ_i)

$\Delta_{i(i \geq 0)}$ are mutually disjoint

Hence there are at most $2^n + 1$ sets, therefore $k \leq 2^n + 1$

Result of the algorithm

When the algorithm terminates:

if $\bot \in \Theta_k$: Γ is unsatisfiable

if $\bot \notin \Theta_k$: Γ is satisfiable, but what does Θ_k represent?

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Result of the algorithm

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- $\Theta_k =$ set of minimum deduction clauses.
- Γ and Θ_k are equivalent.

Property 2.2.5

For all $i < k$, the sets $\Delta_i \cup \Theta_i$ and $\Delta_{i+1} \cup \Theta_{i+1}$ are equivalent.

Hence :

$$\Gamma \equiv \Delta_0 \cup \emptyset = \Delta_0 \cup \Theta_0 \equiv \ldots \equiv \Delta_k \cup \Theta_k = \emptyset \cup \Theta_k = \Theta_k$$

Overview

Today

- Resolution is a correct and complete deductive system: it **characterizes** all the unsatisfiable formulae.
- The DPLL **algorithm** uses ideas from resolution to:
    - find a **model**
    - or else, prove the **unsatisfiability** by an efficient search of the assignments.
- Complete Strategy is an **algorithm** for computing **every** clause deducible from an initial set

Next lecture

- Natural deduction

Homework: **Hypotheses** :

- (H1) : $p \Rightarrow \neg j \equiv \neg p \vee \neg j$
- (H2) : $\neg p \Rightarrow j \equiv p \vee j$
- (H3) : $j \Rightarrow m \equiv \neg j \vee m$
- (¬ C): $\neg m \wedge \neg p$ (two clauses)

Build the proof of $H1, H2, H3, \neg C \vdash \bot$ obtained by the DPLL algorithm (you may pick any variable for branching)