Natural Deduction

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Last course

- Correctness and completeness of resolution
- Complete Strategy
- ► Davis-Putnam algorithm

Homework: solution with DPLL

$$\overline{p} + \overline{j}, p + j, \overline{j} + m, \overline{m}, \overline{p}$$

$$\downarrow \mathsf{RED}$$

$$p + j, \overline{j} + m, \overline{m}, \overline{p}$$

$$\downarrow \mathsf{UR} : \mathsf{m=0}, \mathsf{p=0}$$

$$j, \overline{j}$$

$$\downarrow \mathsf{UR}$$

$$\perp$$

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Plan

Introduction to natural deduction

Rules

Natural deduction proofs

Conclusion

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Intuition

When we write proofs in math courses,

when we decompose a reasoning in elementary obvious steps,

we practice natural deduction.

New deductive systems (1934) introduced by Gentzen (1909-45):

▶ Natural deduction:

- we prove consequences $\Gamma \vdash p$ rather than tautologies
- ▶ only one axiom Γ , $p \vdash p$
- introduction and elimination rules for each connective



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- ▶ only one axiom Γ , $p \vdash p$
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► Sequent calculus:

- ightharpoonup $\Gamma \vdash \Delta$ if whenever all of Γ is true, one of the formulas in Δ is true
- left and right introduction rules

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► Sequent calculus:

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- left and right introduction rules

$$\qquad \textbf{cut rule } \frac{\Gamma \vdash \Delta, p \quad \Gamma', p \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Computing with proofs: cut elimination Every proof that does not use the excluded middle can be transformed into a constructive proof.

Resolution vs. Natural deduction

A proof by **resolution** is a list of clauses built using any of the previous clauses.

In **natural deduction**, during a proof, we can add and remove hypotheses.

T, negation and equivalence are abbreviations defined as:

- ▶ \top abbreviates $\bot \Rightarrow \bot$.
- $ightharpoonup \neg A$ abbreviates $A \Rightarrow \bot$.
- ▶ $A \Leftrightarrow B$ abbreviates $(A \Rightarrow B) \land (B \Rightarrow A)$.

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For example, the formulae $\neg \neg a$, $\neg a \Rightarrow \bot$ and $(a \Rightarrow \bot) \Rightarrow \bot$ are equal.

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Two equal formulae are equivalent!

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Rule

Definition 3.1.1

A rule consists of:

- \blacktriangleright some formulae H_1, \ldots, H_n called **premises** (or hypotheses)
- ► a unique conclusion C
- ► sometimes a name R for the rule

$$\frac{H_1 \dots H_n}{C} R$$

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$$\frac{H_1 \dots H_n}{C} R$$

Example: Proof of a conjunction

$$\frac{A \quad B}{A \wedge B} (\wedge I)$$

Classification of rules

▶ Introduction rules for introducing a connective in the conclusion.

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- ► Elimination rules for removing a connective from one of the premises.

Classification of rules

- ▶ **Introduction rules** for introducing a connective in the conclusion.
- Elimination rules for removing a connective from one of the premises.
- ► + two special rules

Table 3.1

	Introduction	Elimination		
Implication				
_				
Conjunction				
Disjunction				
	Ex falso quodlibet			
\perp				
	Reductio ad absurdum			

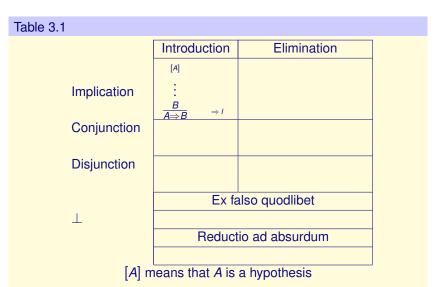
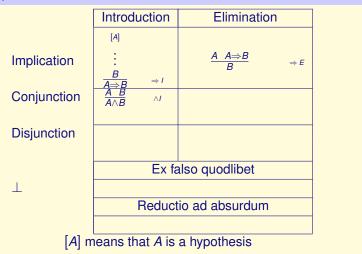
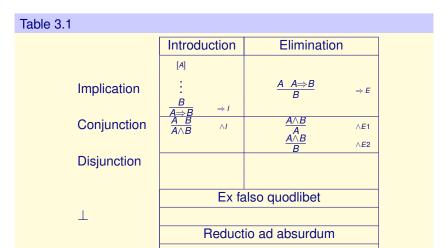


Table 3.1 Introduction Elimination [A] $\frac{A \quad A \Rightarrow B}{B}$ **Implication** $\Rightarrow I$ Conjunction Disjunction Ex falso quodlibet Reductio ad absurdum

[A] means that A is a hypothesis

Table 3.1





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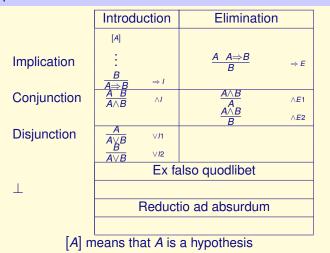


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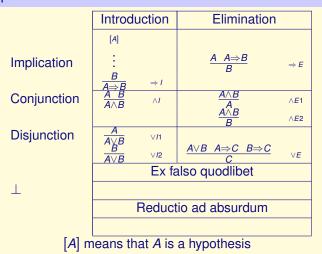


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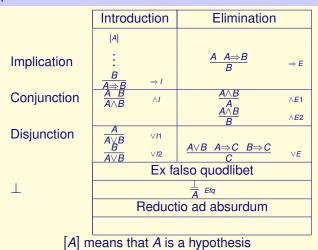


Table 3.1

	Introduction		Elimination		
	[A]				
Implication	:		<u>A A⇒B</u> B	<i>⇒ E</i>	
	<u>B</u> A⇒B	$\Rightarrow I$	В		
Conjunction	$\frac{A \cdot B}{A \wedge B}$	$\wedge I$	$rac{A \wedge B}{A} \ A \wedge B$	∧ <i>E</i> 1	
			$\frac{A \wedge B}{B}$	∧ <i>E</i> 2	
Disjunction	$\frac{A}{A \vee B}$	∨ <i>I</i> 1			
	$\frac{B}{A \vee B}$	∨ <i>I</i> 2	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	∨E	
	Ex falso quodlibet				
	$\frac{\perp}{A}$ Efq				
	Reductio ad absurdum				
	$\frac{\neg \neg A}{A}$ RAA				
[A] means that A is a hypothesis					

$$\frac{A \qquad A \Rightarrow B}{B} \Rightarrow E \qquad \frac{A \qquad A \Rightarrow C}{C} \Rightarrow E$$

$$B \land C \qquad A \Rightarrow C$$

$$\frac{A \qquad A \Rightarrow B}{B} \Rightarrow E \qquad \frac{A \qquad A \Rightarrow C}{C} \Rightarrow E$$

$$B \land C \qquad A \Rightarrow C$$

What have we proven here exactly?

$$\frac{A \qquad A \Rightarrow B}{B} \Rightarrow E \qquad \frac{A \qquad A \Rightarrow C}{C} \Rightarrow E$$

$$B \land C \qquad A \Rightarrow C$$

What have we proven here exactly? $B \wedge C$

$$\frac{A \longrightarrow A \Rightarrow B}{B} \Rightarrow E \qquad \frac{A \longrightarrow C}{C} \Rightarrow E$$

$$B \land C \qquad \land I$$

What have we proven here exactly? $B \wedge C$ under the hypotheses $A, A \Rightarrow B, A \Rightarrow C$

i.e.
$$A, A \Rightarrow B, A \Rightarrow C \models B \land C$$

Fundamental rule of Natural Deduction

Implies-introduction:

In order to prove $A \Rightarrow B$,

just derive *B* with the additional hypothesis *A* and then remove this assumption.

(If
$$A \models B$$
 then $\models A \Rightarrow B$)

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In order to prove $A \Rightarrow B$,

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Definition 3.1.2

A proof line is one of the three following:

- ► Assume formula
- formula
- ► Therefore formula

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A proof line is one of the three following:

- ► Assume formula (to add an hypothesis)
- ► formula (derived from previous lines using the rules)
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Examples:

$$\triangleright$$
 Assume $A \land B$

▶ Therefore
$$A \land B \Rightarrow A$$

$$\frac{[A \land B]}{A} \land E$$

$$A \land B \Rightarrow A \Rightarrow I$$

Proof sketch

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Example 3.1.4

number	line
1	Assume a
2	a∨b
3	Therefore $a \Rightarrow a \lor b$
4	Therefore ¬ a
5	Assume b

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Proof sketch: examples

Where are the sketches?

num	line
1	Assume $a \wedge b$
2	b
3	$b \lor c$
4	Therefore $a \land b \Rightarrow b \lor c$
5	Therefore ¬ a
6	Assume b

num	line
1	Assume a
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num	line
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Context (1/2)

- Each line of a proof sketch has a context
- ► The context is the sequence of hypotheses previously introduced in Assume lines and not removed in Therefore lines.

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Example 3.1.6:

context	number	line	rule
1	1	Assume a	
1,2	2	Assume b	
1,2	3	a∧b	∧I 1,2
1	4	Therefore $b \Rightarrow a \land b$	⇒I 2,3
1,5	5	Assume e	

Context (2/2)

The context of a formula represents the hypotheses from which it has been derived.

Definition 3.1.5

Formally: Γ_i is the context of the line i.

$$\Gamma_0 = \emptyset$$

If the line i is:

- Assume A then $\Gamma_i = \Gamma_{i-1}, i$
- A then $\Gamma_i = \Gamma_{i-1}$
- Therefore A then Γ_i is obtained by deleting the last formula in Γ_{i-1}

Example of context

Write down the **contexts** of the following proof sketch:

context	number	line
	1	Assume a
	2	a∨b
	3	Therefore $a \Rightarrow a \lor b$
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Usable formulae (1/2)

Definition 3.1.7

- ► A formula appearing on a line of a proof sketch is its conclusion.
- ► The conclusion of a line is usable as long as its context (*i.e.*, the hypotheses from which it has been derived) is present.

Usable formulae (1/2)

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Example 3.1.8

context	number	line
1	1	Assume <i>a</i>
1	2	a∨b
	3	Therefore $a \Rightarrow b$
	4	а
	5	b∨a

The conclusion of line 2 is usable on line 2 and not beyond.

Usable formulae (2/2)

On which lines are formulae 1 and 3 usable?

context	number	line
1	1	Assume a
1,2	2 Assume b	
1,2	3	С
1	4	Therefore d
1,5	5	Assume <i>e</i>

Definition of a Proof

Definition 3.1.9

Let Γ be a set of formulae.

A proof in the environment Γ is a proof sketch such that:

- 1. For every "Therefore" line, the formula is $B \Rightarrow C$, where:
 - ► B is the last hypothesis we've removed (from the context of the previous line)
 - ightharpoonup C is either a formula usable on the previous line, or belongs to Γ .

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- 2. For every "A" line, the formula A is:
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- 2. For every "A" line, the formula A is:
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 - ightharpoonup whose premises are usable on the previous line, or belong to Γ.

Beware:

- ▶ The context Γ_i changes during the proof.
- ightharpoonup The environment Γ remains the same.

Proof of formulae

Definition 3.1.10

A proof of formula A within the environment Γ is:

- ightharpoonup either the empty proof (when A is an element of Γ),
- or a proof whose last line is A with an empty context.

Proof of formulae

Definition 3.1.10

A proof of formula A within the environment Γ is:

- ightharpoonup either the empty proof (when A is an element of Γ),
- or a proof whose last line is A with an empty context.

We note:

- ightharpoonup T ightharpoonup A the fact that there is a proof of A within the environment Γ,
- ightharpoonup Γ ightharpoonup : A the fact that P is a proof of A within Γ .
- ▶ When the environment is empty, we abbreviate $\emptyset \vdash A$ by $\vdash A$.
- ▶ When we ask for a proof without indicating the environment, we mean that $\Gamma = \emptyset$.

Let us prove
$$(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$$
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context	number	proof	justification
1	1	Assume $a \Rightarrow b$	

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$$(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$$
.

context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume ¬b	

Let us prove
$$(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$$
.

context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	

Let us prove
$$(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$$
.

context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume ¬b	
1,2,3	3	Assume a	
1,2,3	4	b	\Rightarrow E 1, 3

context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	\Rightarrow E 1, 3
1,2,3	5	上	\Rightarrow E 2, 4

context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	\Rightarrow E 1, 3
1,2,3	5		\Rightarrow E 2, 4
1,2	6	Therefore ¬a	\Rightarrow 13, 5

context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume ¬ b	
1,2,3	3	Assume a	
1,2,3	4	b	\Rightarrow E 1, 3
1,2,3	5		\Rightarrow E 2, 4
1,2	6	Therefore ¬ a	\Rightarrow 13, 5
1	7	Therefore $\neg b \Rightarrow \neg a$	⇒ <i>I</i> 2, 6

context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	⇒ <i>E</i> 1, 3
1,2,3	5		\Rightarrow E 2, 4
1,2	6	Therefore ¬a	⇒ <i>I</i> 3, 5
1	7	Therefore $\neg b \Rightarrow \neg a$	⇒ <i>I</i> 2, 6
	8	Therefore $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$	<i>⇒ I</i> 1,7

Proofs with abbreviations vs. without abbreviations

cont.	n.	proof with abbreviation	proof without abbreviation
1	1	Assume $a \Rightarrow b$	Assume $a \Rightarrow b$
1,2	2	Assume ¬ b	Assume $b \Rightarrow \perp$
1,2,3	3	Assume a	Assume a
1,2,3	4	b	b
1,2,3	5	T	1
1,2	6	Therefore ¬ a	Therefore a⇒⊥
1	7	Therefore $\neg b \Rightarrow \neg a$	Therefore $(b\Rightarrow \perp) \Rightarrow (a\Rightarrow \perp)$
	8	Therefore $(a\Rightarrow b)\Rightarrow (\neg b\Rightarrow \neg a)$	Therefore $(a \Rightarrow b) \Rightarrow ((b \Rightarrow \bot) \Rightarrow (a \Rightarrow \bot))$

Tree (example 3.1.11)

$$\begin{aligned} & \underbrace{\frac{(1)a \Rightarrow \overleftarrow{b} \quad (3)\cancel{a}}{(4)b}}_{} \Rightarrow E \\ & \underbrace{\frac{(5)\bot}{(6)\neg a} \Rightarrow I[3]}_{} \\ & \underbrace{\frac{(7)\neg b \Rightarrow \neg a}{(7)\neg b \Rightarrow \neg a} \Rightarrow I[2]}_{} \\ & \underbrace{(8)(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)}_{} \Rightarrow I[1] \end{aligned}$$

context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume ¬ b	
1,2,3	3	Assume a	
1,2,3	4	Ь	⇒ <i>E</i> 1, 3
1,2,3	5	1	⇒ E 2, 4
1,2	6	Therefore ¬a	⇒ 13,5
1	7	Therefore $\neg b \Rightarrow \neg a$	⇒ <i>I</i> 2, 6
	8	Therefore $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$	⇒ <i>I</i> 1,7

context number proof justification	context	number	proof	justification
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context	number	proof	justification
1	1	Assume $a \land \neg a$	

context	number	proof	justification
1	1	Assume $a \land \neg a$	
1	2	а	<i>∧E</i> 1 1

context	number	proof	justification
1	1	Assume $a \land \neg a$	
1	2	а	<i>∧E</i> 1 1
1	3	$\neg a$	<i>∧E</i> 2 1

context	number	proof justification	
1	1	Assume $a \land \neg a$	
1	2	а	<i>∧E</i> 1 1
1	3	$\neg a$	<i>∧E</i> 2 1
1	4	上	\Rightarrow E 2,3

context	number	proof	justification	
1	1	Assume $a \land \neg a$		
1	2	a	<i>∧E</i> 1 1	
1	3	¬a	<i>∧E</i> 2 1	
1	4		\Rightarrow E 2,3	
1	5	b	\Rightarrow E 2,3 Efq 4	

context	number	proof	justification
1	1	Assume $a \land \neg a$	
1	2	а	<i>∧E</i> 1 1
1	3	$\neg a$	<i>∧E</i> 2 1
1	4		\Rightarrow E 2,3
1	5	b	Efq4
	6	Therefore $a \land \neg a \Rightarrow b$	⇒ <i>I</i> 1,5

Proofs with abbreviations vs. without abbreviation (2/2)

contexte	number	proof with abbreviation	proof without abbreviation	justification
1	1	Assume $a \land \neg a$	Assume $a \land (a \Rightarrow \bot)$	
1	2	а	а	∧ <i>E</i> 1 1
1	3	$\neg a$	a⇒⊥	∧ <i>E</i> 2 1
1	4	1	<u> </u>	\Rightarrow E 2,3
1	5	b	b	Efq4
	6	Therefore $a \land \neg a \Rightarrow b$	Therefore $a \land (a \Rightarrow \bot) \Rightarrow b$	⇒ <i>I</i> 1,5

Plan

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Today

- Propositional natural deduction reflects the usual deduction rules into a formal system.
- Unlike in resolution, a proof occurs in a context (list of formulae assumed at a given point).

Next lecture

- Completeness
- Correctness
- Tactics

Homework: prove

$$(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p$$

using natural deduction.