Natural Deduction

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Last course

- Correctness and completeness of resolution
- Complete Strategy
- Davis-Putnam algorithm

Homework: solution with DPLL

$$\overline{p}+\bar{j}, p+j, \bar{j}+m, \overline{m}, \overline{p}$$

$$\Big\downarrow \text{RED}$$

$$p+j, \bar{j}+m, \overline{m}, \overline{p}$$

$$\Big\downarrow \text{UR : m=0, p=0}$$

$$j, \bar{j}$$

$$\Big\downarrow \text{UR}$$

$$\bot$$

Plan

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Intuition

When we write proofs in math courses,

when we decompose a reasoning in elementary obvious steps,

we practice natural deduction.

Natural Deduction (ND)

New deductive systems (1934) introduced by Gentzen (1909-45):

- **Natural deduction**:
  - we prove consequences $\Gamma \vdash p$ rather than tautologies
  - only one axiom $\Gamma, p \vdash p$
  - introduction and elimination rules for each connective

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- **Sequent calculus**:
  - $\Gamma \vdash \Delta$ if whenever all of Γ is true, one of the formulas in Δ is true
  - left and right introduction rules
  - *cut* rule $\dfrac{\Gamma \vdash \Delta, p \quad \Gamma', p \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$

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  - left and right introduction rules
  - *cut* rule $\dfrac{\Gamma \vdash \Delta, p \quad \Gamma', p \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$
- Computing with proofs: **cut elimination**
  Every proof that does not use the excluded middle can be transformed into a **constructive** proof.

Resolution *vs.* Natural deduction

A proof by **resolution** is a list of clauses built using any of the previous clauses.

In **natural deduction**, during a proof,
we can add and remove hypotheses.

Abbreviations

$\top$, negation and equivalence are abbreviations defined as:

- $\top$ abbreviates $\bot \Rightarrow \bot$.
- $\neg A$ abbreviates $A \Rightarrow \bot$.
- $A \Leftrightarrow B$ abbreviates $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

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Two equal formulae are equivalent!

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Rule

Definition 3.1.1

A rule consists of:

- some formulae $H_1, \ldots, H_n$ called **premises** (or hypotheses)
- a unique **conclusion** C
- sometimes a name R for the rule

$$\frac{H_1 \ldots H_n}{C}\ R$$

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$$\frac{H_1 \ldots H_n}{C}\ R$$

Example : Proof of a conjunction

$$\frac{A \quad B}{A \wedge B}\ (\wedge I)$$

Classification of rules

- **Introduction rules** for introducing a connective in the conclusion.

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- $+$ **two special rules**

The rules (system NK of Gentzen)

Table 3.1

	Introduction	Elimination
Implication		
Conjunction		
Disjunction		
	Ex falso quodlibet	
$\perp$		
	Reductio ad absurdum	

The rules (system NK of Gentzen)

Table 3.1

	Introduction	Elimination
Implication	$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow I$	
Conjunction		
Disjunction		
$\perp$	Ex falso quodlibet	
	Reductio ad absurdum	

$[A]$ means that A is a hypothesis

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Conjunction	$\dfrac{A \quad B}{A\wedge B}\ \wedge I$	$\dfrac{A\wedge B}{A}\ \wedge E1$ $\dfrac{A\wedge B}{B}\ \wedge E2$
Disjunction		
	Ex falso quodlibet	
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Conjunction	$\dfrac{A \quad B}{A \wedge B} \wedge I$	$\dfrac{A \wedge B}{A} \wedge E1$ $\dfrac{A \wedge B}{B} \wedge E2$
Disjunction	$\dfrac{A}{A \vee B} \vee I1$ $\dfrac{B}{A \vee B} \vee I2$	
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Disjunction	$\dfrac{A}{A \vee B} \vee I1$ $\dfrac{B}{A \vee B} \vee I2$	$\dfrac{A \vee B \quad A \Rightarrow C \quad B \Rightarrow C}{C} \vee E$
$\perp$	Ex falso quodlibet	
	Reductio ad absurdum	

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	Introduction	Elimination
Implication	$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow I$	$\dfrac{A \quad A \Rightarrow B}{B} \Rightarrow E$
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Disjunction	$\dfrac{A}{A \vee B} \vee I1$ $\dfrac{B}{A \vee B} \vee I2$	$\dfrac{A \vee B \quad A \Rightarrow C \quad B \Rightarrow C}{C} \vee E$
	Ex falso quodlibet	
$\bot$	$\dfrac{\bot}{A}$ *Efq*	
	Reductio ad absurdum	

[*A*] means that *A* is a hypothesis

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$\perp$	Ex falso quodlibet: $\dfrac{\perp}{A} Efq$	
	Reductio ad absurdum: $\dfrac{\neg\neg A}{A} RAA$	

$[A]$ means that A is a hypothesis

A “simple” example

$$\dfrac{\dfrac{A \qquad A \Rightarrow B}{B} \Rightarrow E \qquad \dfrac{A \qquad A \Rightarrow C}{C} \Rightarrow E}{B \wedge C} \wedge I$$

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What have we proven here exactly?

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What have we proven here exactly? $B \wedge C$

A "simple" example

$$\dfrac{\dfrac{A \qquad A \Rightarrow B}{B} \Rightarrow E \qquad \dfrac{A \qquad A \Rightarrow C}{C} \Rightarrow E}{B \wedge C} \wedge I$$

What have we proven here exactly? $B \wedge C$
under the hypotheses $A,\ A \Rightarrow B,\ A \Rightarrow C$

i.e. $A,\ A \Rightarrow B,\ A \Rightarrow C \vDash B \wedge C$

Fundamental rule of Natural Deduction

Implies-introduction:

In order to prove $A \Rightarrow B$,
just derive B with the additional hypothesis A and then remove this assumption.

$$(\text{If } A \models B \text{ then } \models A \Rightarrow B)$$

Fundamental rule of Natural Deduction

Implies-introduction:

In order to prove $A \Rightarrow B$,
just derive B with the additional hypothesis A and then remove this assumption.

(If $A \models B$ then $\models A \Rightarrow B$)

$$\dfrac{\begin{array}{ccccc} [A] & H_1 & \dots & H_n \\ & \ddots & \vdots & \iddots \\ & & B & \end{array}}{A \Rightarrow B} \Rightarrow I$$

proves that $H_1, \dots, H_n \models A \Rightarrow B$.

Plan

Proof line

Definition 3.1.2

A proof line is one of the three following:

- `Assume` formula
- formula
- `Therefore` formula

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This last case is the rule of implies-introduction.

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- formula (derived from previous lines using the rules)
- `Therefore` formula (to remove the last hypothesis)

This last case is the rule of implies-introduction.

Examples:

- `Assume` $A \wedge B$
- A
- `Therefore` $A \wedge B \Rightarrow A$

$$\dfrac{\dfrac{[A \wedge B]}{A}\,\wedge E}{A \wedge B \Rightarrow A}\,\Rightarrow I$$

Proof sketch

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A proof sketch is a sequence of lines such that, in every prefix of the sequence, there are at least as many `Assume` as `Therefore`.

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Example 3.1.4

number	line
1	Assume a
2	$a \vee b$
3	Therefore $a \Rightarrow a \vee b$
4	Therefore $\neg a$
5	Assume b

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Proof sketch: examples

Where are the sketches?

num	line
1	Assume $a \wedge b$
2	b
3	$b \vee c$
4	Therefore $a \wedge b \Rightarrow b \vee c$
5	Therefore $\neg a$
6	Assume b

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Context (1/2)

- Each line of a proof sketch has a context
- The context is the sequence of hypotheses previously introduced in `Assume` lines and not removed in `Therefore` lines.

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Example 3.1.6:

context	number	line	rule
1	1	`Assume` a	
1,2	2	`Assume` b	
1,2	3	$a \wedge b$	$\wedge$I 1,2
1	4	`Therefore` $b \Rightarrow a \wedge b$	$\Rightarrow$I 2,3
1,5	5	`Assume` e	

Context (2/2)

The context of a formula represents the hypotheses from which it has been derived.

Definition 3.1.5

Formally: Γ_i is the context of the line i.

$\Gamma_0 = \emptyset$

If the line i is:

- `Assume` A
  then $\Gamma_i = \Gamma_{i-1}, i$
- A
  then $\Gamma_i = \Gamma_{i-1}$
- `Therefore` A
  then Γ_i is obtained by deleting the last formula in Γ_{i-1}

Example of context

Write down the **contexts** of the following proof sketch:

context	number	line
	1	`Assume` a
	2	$a \vee b$
	3	`Therefore` $a \Rightarrow a \vee b$
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	5	Therefore b

Usable formulae (1/2)

Definition 3.1.7

- A formula appearing on a line of a proof sketch is its conclusion.
- The conclusion of a line is usable as long as its context (*i.e.*, the hypotheses from which it has been derived) is present.

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Example 3.1.8

context	number	line
1	1	Assume a
1	2	$a \vee b$
	3	Therefore $a \Rightarrow b$
	4	a
	5	$b \vee a$

The conclusion of line 2 is usable on line 2 and not beyond.

Usable formulae (2/2)

On which lines are formulae 1 and 3 **usable**?

context	number	line
1	1	`Assume` a
1,2	2	`Assume` b
1,2	3	c
1	4	`Therefore` d
1,5	5	`Assume` e

Definition of a Proof

Definition 3.1.9

Let Γ be a set of formulae.
A proof in the environment Γ is a proof sketch such that:

1. For every “`Therefore`” line, the formula is $B \Rightarrow C$, where:
   - B is the last hypothesis we've removed (from the context of the previous line)
   - C is either a formula usable on the previous line, or belongs to Γ.

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2. For every "A" line, the formula A is:
   - the conclusion of a rule (other than $\Rightarrow I$)
   - whose premises are usable on the previous line, or belong to Γ.

Beware:

- The context Γ_i changes during the proof.
- The environment Γ remains the same.

Proof of formulae

Definition 3.1.10

A proof of formula A within the environment Γ is:

- either the empty proof (when A is an element of Γ),
- or a proof whose last line is A with an empty context.

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- or a proof whose last line is A with an empty context.

We note:

- $\Gamma \vdash A$ the fact that there is a proof of A within the environment Γ,
- $\Gamma \vdash P : A$ the fact that P is a proof of A within Γ.
- When the environment is empty, we abbreviate $\emptyset \vdash A$ by $\vdash A$.
- When we ask for a proof without indicating the environment, we mean that $\Gamma = \emptyset$.

First Example (exemple 3.1.11)

Let us prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$.

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1,2	2	Assume $\neg b$	

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1,2,3	3	Assume a	

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1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3

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1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\bot$	$\Rightarrow E$ 2, 4

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1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\perp$	$\Rightarrow E$ 2, 4
1,2	6	Therefore $\neg a$	$\Rightarrow I$ 3, 5

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1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\bot$	$\Rightarrow E$ 2, 4
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1	7	Therefore $\neg b \Rightarrow \neg a$	$\Rightarrow I$ 2, 6

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1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\perp$	$\Rightarrow E$ 2, 4
1,2	6	Therefore $\neg a$	$\Rightarrow I$ 3, 5
1	7	Therefore $\neg b \Rightarrow \neg a$	$\Rightarrow I$ 2, 6
	8	Therefore $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$	$\Rightarrow I$ 1,7

Proofs with abbreviations *vs.* without abbreviations

cont.	n.	proof with abbreviation	proof without abbreviation
1	1	Assume $a \Rightarrow b$	Assume $a \Rightarrow b$
1,2	2	Assume $\neg b$	Assume $b \Rightarrow \bot$
1,2,3	3	Assume a	Assume a
1,2,3	4	b	b
1,2,3	5	$\bot$	$\bot$
1,2	6	Therefore $\neg a$	Therefore $a \Rightarrow \bot$
1	7	Therefore $\neg b \Rightarrow \neg a$	Therefore $(b \Rightarrow \bot) \Rightarrow (a \Rightarrow \bot)$
	8	Therefore $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$	Therefore $(a \Rightarrow b) \Rightarrow ((b \Rightarrow \bot) \Rightarrow (a \Rightarrow \bot))$

Tree (example 3.1.11)

$$\dfrac{\dfrac{\dfrac{\dfrac{(2)\cancel{\neg b} \qquad \dfrac{(1)\cancel{a \Rightarrow b} \quad (3)\cancel{a}}{(4)b} \Rightarrow E}{(5)\bot} \Rightarrow E}{(6)\neg a} \Rightarrow I[3]}{(7)\neg b \Rightarrow \neg a} \Rightarrow I[2]}{(8)(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)} \Rightarrow I[1]$$

context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\bot$	$\Rightarrow E$ 2, 4
1,2	6	Therefore $\neg a$	$\Rightarrow I$ 3, 5
1	7	Therefore $\neg b \Rightarrow \neg a$	$\Rightarrow I$ 2, 6
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Second Example

Prove $a \wedge \neg a \Rightarrow b$.

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1	1	Assume $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
1	3	$\neg a$	$\wedge E2$ 1

Second Example

Prove $a \wedge \neg a \Rightarrow b$.

context	number	proof	justification
1	1	$\texttt{Assume}\ a \wedge \neg a$	
1	2	a	$\wedge E1\ 1$
1	3	$\neg a$	$\wedge E2\ 1$
1	4	$\bot$	$\Rightarrow E\ 2,3$

Second Example

Prove $a \wedge \neg a \Rightarrow b$.

context	number	proof	justification
1	1	Assume $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
1	3	$\neg a$	$\wedge E2$ 1
1	4	$\perp$	$\Rightarrow E$ 2, 3
1	5	b	Efq 4

Second Example

Prove $a \wedge \neg a \Rightarrow b$.

context	number	proof	justification
1	1	Assume $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
1	3	$\neg a$	$\wedge E2$ 1
1	4	$\bot$	$\Rightarrow E$ 2, 3
1	5	b	$Efq4$
	6	Therefore $a \wedge \neg a \Rightarrow b$	$\Rightarrow I$ 1, 5

Proofs with abbreviations *vs.* without abbreviation (2/2)

contexte	number	proof with abbreviation	proof without abbreviation	justification
1	1	Assume $a \land \neg a$	Assume $a \land (a \Rightarrow \bot)$	
1	2	a	a	$\land E1$ 1
1	3	$\neg a$	$a \Rightarrow \bot$	$\land E2$ 1
1	4	$\bot$	$\bot$	$\Rightarrow E$ 2,3
1	5	b	b	*Efq*4
	6	Therefore $a \land \neg a \Rightarrow b$	Therefore $a \land (a \Rightarrow \bot) \Rightarrow b$	$\Rightarrow I$ 1,5

Plan

Today

- Propositional natural deduction reflects the usual **deduction rules** into a formal system.
- Unlike in resolution, a proof occurs in a context (list of formulae **assumed** at a given point).

Next lecture

- Completeness
- Correctness
- Tactics

Homework: prove

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

using natural deduction.