Natural Deduction

Properties and tactics

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Last lecture

Natural deduction

- Rules
- Context
- Proofs

Reminder of the rules

	Introduction	Elimination
Implication	$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow I$	$\dfrac{A \quad A \Rightarrow B}{B} \Rightarrow E$
Conjunction	$\dfrac{A \quad B}{A \wedge B} \wedge I$	$\dfrac{A \wedge B}{A} \wedge E1$ $\dfrac{A \wedge B}{B} \wedge E2$
Disjunction	$\dfrac{A}{A \vee B} \vee I1$ $\dfrac{B}{A \vee B} \vee I2$	$\dfrac{A \vee B \quad A \Rightarrow C \quad B \Rightarrow C}{C} \vee E$
	Ex falso quodlibet	
$\perp$	$\dfrac{\perp}{A}$ *Efq*	
	Reductio ad absurdum	
	$\dfrac{\neg\neg A}{A}$ *RAA*	

Second Example

Prove $a \land \neg a \Rightarrow b$.

context	number	proof	justification

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Prove $a \wedge \neg a \Rightarrow b$.

context	number	proof	justification
1	1	Assume $a \wedge \neg a$	

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1	1	Assume $a \wedge \neg a$	
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context	number	proof	justification
1	1	Assume $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
1	3	$\neg a$	$\wedge E2$ 1

Second Example

Prove $a \wedge \neg a \Rightarrow b$.

context	number	proof	justification
1	1	Assume $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
1	3	$\neg a$	$\wedge E2$ 1
1	4	$\bot$	$\Rightarrow E$ 2, 3

Second Example

Prove $a \wedge \neg a \Rightarrow b$.

context	number	proof	justification
1	1	$\texttt{Assume}\ a \wedge \neg a$	
1	2	a	$\wedge E1\ 1$
1	3	$\neg a$	$\wedge E2\ 1$
1	4	$\bot$	$\Rightarrow E\ 2,3$
1	5	b	$Efq\ 4$

Second Example

Prove $a \wedge \neg a \Rightarrow b$.

context	number	proof	justification
1	1	Assume $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
1	3	$\neg a$	$\wedge E2$ 1
1	4	$\bot$	$\Rightarrow E$ 2,3
1	5	b	Efq4
	6	Therefore $a \wedge \neg a \Rightarrow b$	$\Rightarrow I$ 1,5

Third Example: with an environment

Prove $\neg A$ in the environment $\neg(A \vee B)$

environment			
reference		formula	
(i)		$\neg(A \vee B)$	
context	number	proof	justification

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1	1	`Assume` A	

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(i)		$\neg(A \vee B)$	
context	number	proof	justification
1	1	Assume A	
1	2	$A \vee B$	$\vee I1$ 1

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rference		formula	
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context	number	proof	justification
1	1	Assume A	
1	2	$A \vee B$	$\vee I1$ 1
1	3	$\perp$	$\Rightarrow E\ i, 2$

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reference		formula	
(i)		$\neg(A \vee B)$	
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1	1	Assume A	
1	2	$A \vee B$	$\vee I1$ 1
1	3	$\perp$	$\Rightarrow E\ i, 2$
	4	Therefore $\neg A$	$\Rightarrow I\ 1, 3$

Fourth exemple (example 3.1.12)

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$.

environment			
reference		formule	
(i)		$A \Rightarrow B$	
context	number	proof	justification

Fourth exemple (example 3.1.12)

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	proof	justification
1	1	`Assume` $\neg(\neg A \vee B)$	

Fourth exemple (example 3.1.12)

Prove $\neg A \lor B$ in the environment $A \Rightarrow B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	proof	justification
1	1	`Assume` $\neg(\neg A \lor B)$	
1,2	2	`Assume` A	

Fourth exemple (example 3.1.12)

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	proof	justification
1	1	`Assume` $\neg(\neg A \vee B)$	
1,2	2	`Assume` A	
1,2	3	B	$\Rightarrow E\ i, 2$

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Prove $\neg A \vee B$ in the environment $A \Rightarrow B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	proof	justification
1	1	Assume $\neg(\neg A \vee B)$	
1,2	2	Assume A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$

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Prove $\neg A \vee B$ in the environment $A \Rightarrow B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	proof	justification
1	1	`Assume` $\neg(\neg A \vee B)$	
1,2	2	`Assume` A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$
1,2	5	$\perp$	$\Rightarrow E\ 1, 4$

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reference		formula	
(i)		$A \Rightarrow B$	
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1	1	Assume $\neg(\neg A \vee B)$	
1,2	2	Assume A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$
1,2	5	$\perp$	$\Rightarrow E\ 1, 4$
1	6	Therefore $\neg A$	$\Rightarrow I\ 2, 5$

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reference		formula	
(i)		$A \Rightarrow B$	
context	number	proof	justification
1	1	`Assume` $\neg(\neg A \vee B)$	
1,2	2	`Assume` A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$
1,2	5	$\perp$	$\Rightarrow E\ 1, 4$
1	6	`Therefore` $\neg A$	$\Rightarrow I\ 2, 5$
1	7	$\neg A \vee B$	$\vee I1\ 6$

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1	1	`Assume` $\neg(\neg A \vee B)$	
1,2	2	`Assume` A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$
1,2	5	$\bot$	$\Rightarrow E\ 1, 4$
1	6	`Therefore` $\neg A$	$\Rightarrow I\ 2, 5$
1	7	$\neg A \vee B$	$\vee I1\ 6$
1	8	$\bot$	$\Rightarrow E\ 1, 7$

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(i)		$A \Rightarrow B$	
context	number	proof	justification
1	1	Assume $\neg(\neg A \vee B)$	
1,2	2	Assume A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$
1,2	5	$\perp$	$\Rightarrow E\ 1, 4$
1	6	Therefore $\neg A$	$\Rightarrow I\ 2, 5$
1	7	$\neg A \vee B$	$\vee I1\ 6$
1	8	$\perp$	$\Rightarrow E\ 1, 7$
	9	Therefore $\neg\neg(\neg A \vee B)$	$\Rightarrow I\ 1, 8$

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(i)		$A \Rightarrow B$	
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1	1	Assume $\neg(\neg A \vee B)$	
1,2	2	Assume A	
1,2	3	B	$\Rightarrow E\ i, 2$
1,2	4	$\neg A \vee B$	$\vee I2\ 3$
1,2	5	$\bot$	$\Rightarrow E\ 1, 4$
1	6	Therefore $\neg A$	$\Rightarrow I\ 2, 5$
1	7	$\neg A \vee B$	$\vee I1\ 6$
1	8	$\bot$	$\Rightarrow E\ 1, 7$
	9	Therefore $\neg\neg(\neg A \vee B)$	$\Rightarrow I\ 1, 8$
	10	$\neg A \vee B$	$RAA\ 9$

Tree (example 3.1.12)

Give the tree representation of the previous proof:

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$$
\dfrac{\dfrac{\dfrac{(1)\cancel{\neg(\neg A \vee B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(1)\cancel{\neg(\neg A \vee B)} \qquad \dfrac{\dfrac{(i)A \Rightarrow B \quad (2)\cancel{A}}{(3)B}\Rightarrow E}{(4)\neg A \vee B}\vee 2}{(5)\bot}\Rightarrow E}{(6)\neg A}\Rightarrow I[2]}{(7)\neg A \vee B}\vee 1}{(8)\bot}\Rightarrow E}{(9)\neg\neg(\neg A \vee B)}\Rightarrow I[1]}{(10)\neg A \vee B} RAA
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$$

The environment consists of formulae occurring at non-removed leaves.

Intuitionism and constructivism (Brouwer, 1881-1966)

In the wake of Poincaré, he founded (in 1918) the **intuitionist** philosophy: the validity of mathematics should be verifiable by the human mind.

- refusal of infinite objects such as the ones of set theory
- in particular, notion of **constructible real number** = algorithm that produces its digits

Example of a non-constructive proof : assume $P(0)$ and $\neg P(2)$. Then $\exists x(P(x) \wedge \neg P(x+1))$

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The introduction rules for $\vee$ make it explicit which case is true: following the reasoning step-by-step is an *algorithm*!

However the rule $\dfrac{\neg\neg A}{A}$ allows to override that constraint.

Our running example

context	number	proof	justification
1	1	Assume $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$\neg p \Rightarrow j$	$\wedge E$ 1
1	3	$j \Rightarrow m$	$\wedge E$ 1
1,4	4	Assume $\neg(m \vee p)$	
1,4,5	5	Assume p	
1,4,5	6	$m \vee p$	$\vee I$ 5
1,4,5	7	$\perp$	$\Rightarrow E$ 4,6
1,4	8	Therefore $\neg p$	$\Rightarrow I$ 5,7
1,4	9	j	$\Rightarrow E$ 2, 8
1,4	10	m	$\Rightarrow E$ 3, 9
1,4	11	$m \vee p$	$\vee I$ 10
1,4	12	$\perp$	$\Rightarrow E$ 4, 11
1	13	Therefore $\neg\neg(m \vee p)$	$\Rightarrow I$ 4, 13
1	14	$m \vee p$	RAA 13

Plan

Theorem

Theorem 3.3.1

If a formula A is deduced from an environment Γ ($\Gamma \vdash A$) then A is a consequence of Γ ($\Gamma \models A$).

Every proof written in an environment Γ is correct!

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Proof by induction on the number of lines in the proof P:

- Let H_i be the context and C_i the conclusion of the i^{th} line in P.
- We show that for every k we have $\Gamma, H_k \models C_k$.

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Proof by induction on the number of lines in the proof P:

- Let H_i be the context and C_i the conclusion of the i^{th} line in P.
- We show that for every k we have $\Gamma, H_k \models C_k$.

Hence, for the last line n of the proof: $\Gamma \models A$
(H_n is empty and $C_n = A$)

Base case

Assume that A is derived from Γ by an empty proof.

That is, A is a member of Γ.

Hence $\Gamma \models A$.

Induction hypothesis

Assume that for every line $i < k$ of the proof we have $\Gamma,\ H_i \models C_i$.

Let us prove that $\Gamma,\ H_k \models C_k$.

Induction hypothesis

Assume that for every line $i < k$ of the proof we have $\Gamma, \ H_i \models C_i$.

Let us prove that $\Gamma, \ H_k \models C_k$.

Three possible cases:

- Line k is "`Assume` C_k".
- Line k is "`Therefore` C_k".
- Line k is "C_k".

Line k is "`Assume` C_k"

The formula C_k is the last formula of H_k.

Then $\Gamma, H_k \models C_k$.

The line k is "`Therefore` C_k"

C_k is the formula $B \Rightarrow D$ where:

- B is the last formula of H_{k-1} and $H_{k-1} = H_k, B$
- D is usable on the previous line.

The line k is "`Therefore` C_k"

C_k is the formula $B \Rightarrow D$ where:

- B is the last formula of H_{k-1} and $H_{k-1} = H_k, B$
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Hence there exists a line $i < k$ such that $D = C_i$ and H_i is a prefix of H_{k-1}.
By induction hypothesis, $\Gamma,\ H_i \models D$.

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Hence there exists a line $i < k$ such that $D = C_i$ and H_i is a prefix of H_{k-1}.
By induction hypothesis, $\Gamma,\ H_i \models D$.

Since H_i is a prefix of H_{k-1}, we have $\Gamma,\ H_{k-1} \models D$
which can also be written $\Gamma,\ H_k,\ B \models D$.
Therefore $\Gamma,\ H_k \models B \Rightarrow D$.

Line k is "C_k"

C_k is then the conclusion of a rule, whose premises either:

- are usable on the previous line
- or belong to Γ.

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We only consider the rule $\wedge$I, the other cases being similar.
$C_k = (D \wedge E)$ and the premises of the rule are D and E.

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By induction hypothesis, we have:
$\Gamma,\ H_{k-1} \models D$ and $\Gamma,\ H_{k-1} \models E$.
Since the line k does not change the hypotheses, we have $H_{k-1} = H_k$.

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Finally $D, E \models D \wedge E$. Transitively, $\Gamma, H_k \models C_k$.

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Finally $D, E \models D \wedge E$. Transitively, $\Gamma, H_k \models C_k$.

For the other rules, it is the same proof, you just have to prove that the conclusion is a consequence of the premises.

Plan

Correctness

Completeness

Tactics

Conclusion

Theorem

We prove the completeness of the rules only for formulas containing the following logic symbols: $\bot$, $\wedge$, $\vee$, $\Rightarrow$.

This is enough because additional symbols $\top$, $\neg$ and $\Leftrightarrow$ can be regarded as abbreviations.

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Theorem 3.4.1

Let Γ be a finite set of formulae and A a formula.
If $\Gamma \models A$ then $\Gamma \vdash A$.

Definitions

A literal is either a variable x or an implication $x \Rightarrow \bot$.
x and $x \Rightarrow \bot$ (abbreviated as $\neg x$) are complementary literals.

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We define a measure m of formulae and of lists of formulae as:

- $m(\bot) = 0$
- $m(x) = 1$
- $m(A \Rightarrow B) = 1 + m(A) + m(B)$
- $m(A \wedge B) = 1 + m(A) + m(B)$
- $m(A \vee B) = 2 + m(A) + m(B)$
- $m(\Gamma) = \sum_{A \in \Gamma} m(A)$

Definitions

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We define a measure m of formulae and of lists of formulae as:

- $m(\perp) = 0$
- $m(x) = 1$
- $m(A \Rightarrow B) = 1 + m(A) + m(B)$ (thus $m(\neg A) = m(A) + 1$)
- $m(A \wedge B) = 1 + m(A) + m(B)$
- $m(A \vee B) = 2 + m(A) + m(B)$
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- $m(A \vee B) = 2 + m(A) + m(B)$
- $m(\Gamma) = \sum_{A \in \Gamma} m(A)$

For example, let $A = (a \vee \neg a)$.
$m(\neg a) = 2$, $m(A) = 5$ and $m(A,\ (b \wedge b),\ A) = 13$.

Induction

We define $P(n)$ to be the following property:
If $m(\Gamma,\ A) = n$, then if $\Gamma \models A$ then $\Gamma \vdash A$.

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Induction

We define $P(n)$ to be the following property:
If $m(\Gamma, A) = n$, then if $\Gamma \models A$ then $\Gamma \vdash A$.

To show that $P(n)$ holds for every integer n, we use "strong" induction:

Assume that for every $i < k$, property $P(i)$ holds.
Assume that $m(\Gamma, A) = k$ and $\Gamma \models A$.
Let us show that $\Gamma \vdash A$.

Decomposition

Idea: we decompose Γ, A in order to apply the induction hypothesis.

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We study three cases:

Case 1: Neither A, nor Γ is decomposable.

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Case 1: Neither A, nor Γ is decomposable.

Case 2: A is decomposable.
We decompose A in two sub-formulae B and C.
We obtain $m(\Gamma,\ B) < m(\Gamma,\ A)$ and $m(\Gamma,\ C) < m(\Gamma,\ A)$.

Decomposition

Idea: we decompose Γ, A in order to apply the induction hypothesis.

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Case 1: Neither A, nor Γ is decomposable.

Case 2: A is decomposable.
We decompose A in two sub-formulae B and C.
We obtain $m(\Gamma,\ B) < m(\Gamma,\ A)$ and $m(\Gamma,\ C) < m(\Gamma,\ A)$.

Case 3: Γ is decomposable.
We choose a decomposable formula (other than $x \Rightarrow \bot$) in Γ.

Case 1 : neither A, nor Γ are decomposable

Then:

- Γ is a list of literals or contains the formula $\bot$.
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(a) If $\bot \in \Gamma$ then A can be derived from $\bot$ by the rule *Efq*.

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(a) If $\bot \in \Gamma$ then A can be derived from $\bot$ by the rule *Efq*.
(b) If Γ is a list of literals then we have two cases:

Case 1 : neither A, nor Γ are decomposable

Then:

- Γ is a list of literals or contains the formula $\perp$.
- A is $\perp$ or a variable.

(a) If $\perp \in \Gamma$ then A can be derived from $\perp$ by the rule *Efq*.

(b) If Γ is a list of literals then we have two cases:

- $A = \perp$.
  Since $s(\Gamma) \models A$, there are two complementary literals in Γ.
  Therefore A can be derived from Γ by the rule $\Rightarrow$E.

Case 1 : neither A, nor Γ are decomposable

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  - $A = \bot$.
    Since $s(\Gamma) \models A$, there are two complementary literals in Γ.
    Therefore A can be derived from Γ by the rule $\Rightarrow$E.
  - A is a variable.
    Since $\Gamma \models A$:

Case 1 : neither A, nor Γ are decomposable

Then:

- Γ is a list of literals or contains the formula $\bot$.
- A is $\bot$ or a variable.

(a) If $\bot \in \Gamma$ then A can be derived from $\bot$ by the rule *Efq*.

(b) If Γ is a list of literals then we have two cases:

  - $A = \bot$.
    Since $s(\Gamma) \models A$, there are two complementary literals in Γ.
    Therefore A can be derived from Γ by the rule $\Rightarrow$E.
  - A is a variable.
    Since $\Gamma \models A$:
    - either Γ contains two complementary literals, and similarly $\Gamma \vdash A$

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    Therefore A can be derived from Γ by the rule $\Rightarrow$E.
  - A is a variable.
    Since $\Gamma \models A$:
    - either Γ contains two complementary literals, and similarly $\Gamma \vdash A$
    - or $A \in \Gamma$ and in this case $\Gamma \vdash A$ (by empty proof).

Case 2: A is decomposable into B and C

A is decomposed into $B \wedge C$, $B \vee C$, or $B \Rightarrow C$.

We only study the case $A = B \wedge C$, the other cases are similar.

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By induction hypothesis, there exist two proofs P and Q such that $\Gamma \vdash P : B$ and $\Gamma \vdash Q : C$.

Hence the proof "$P,\ Q,\ A$" is a proof of A in the environment Γ.

Case 3: Γ is decomposable

There is a decomposable formula in Γ which is either:

- $B \wedge C$
- $B \vee C$
- $B \Rightarrow C$ où $C \neq \bot$
- $(B \wedge C) \Rightarrow \bot$
- $(B \vee C) \Rightarrow \bot$
- $(B \Rightarrow C) \Rightarrow \bot$

We only study the first case.

Γ is a permutation of the list $(B \wedge C), \Delta$

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Since $\Gamma \models A$, we have $B, C, \Delta \models A$.

The sum of the measures of B and C is strictly less than the measure of $B \wedge C$.

Hence $m(B, C, \Delta, A) < m((B \wedge C), \Delta, A) = m(\Gamma, A) = k$, by induction hypothesis, there exist a proof P such that $B, C, \Delta \vdash P : A$.

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Hence $m(B, C, \Delta, A) < m((B \wedge C), \Delta, A) = m(\Gamma, A) = k$, by induction hypothesis, there exist a proof P such that $B, C, \Delta \vdash P : A$.

Since B can be derived from $(B \wedge C)$ by the rule $\wedge$E1 and C can be derived from $(B \wedge C)$ by the rule $\wedge$E2 : "$B,\ C,\ P$" is a proof of A in the environment Γ.

Plan

Correctness

Completeness

Tactics

Conclusion

Remark 3.4.2

The proof of completeness is constructive, that is it provides an algorithm to build a proof of a formula in an environment.

However, this algorithm can lead to long proofs.

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The proof of completeness is constructive, that is it provides an algorithm to build a proof of a formula in an environment.

However, this algorithm can lead to long proofs.

The tool
http://teachinglogic.univ-grenoble-alpes.fr/DN/
builds proofs more more efficiently.
It uses "optimised" tactics presented in section 3.2.

For example, to prove $B \vee C$:

- First try to prove B
- If failure, then try to prove C
- Otherwise, use tactic 10 (prove C under the hypothesis $\neg B$)

Proof tactics

We wish to prove A in the environment Γ

The 13 following tactics must be used in the following order!

- Tactics 1 to 3 : the proof is over
- Tactics 4 to 6 : proof guided by the conclusion (Intro rules)
- Tactics 7 to 9 : proof guided by the environment (Elim rules)
- Tactics 10 to 13 : reasoning by absurd

Tactic 1

If $A \in \Gamma$ then the empty proof is obtained.

Tactic 2

If A is the consequence of a rule whose premises are in Γ, then the obtained proof is
"A".

Tactic 3

If Γ contains a contradiction, that is a formula B and a formula $\neg B$, then the obtained proof is "$\bot$, A".

Tactic 4

If A is $B \wedge C$ then:

contexte	preuve	justification
Γ	B	$\cdots P \cdots$
Γ	C	$\cdots Q \cdots$
Γ	$B \wedge C$	$\wedge I$

Tactic 4

If A is $B \wedge C$ then:

contexte	preuve	justification
Γ	B	$\cdots P \cdots$
Γ	C	$\cdots Q \cdots$
Γ	$B \wedge C$	$\wedge I$

The proofs can fail (if $\Gamma \not\models A$).
Here, if the proof of B or C fails, the proof of A fails too.

In the remainder of the lecture, we do not highlight the failure cases anymore, unless another proof has to be tried.

Tactic 5

If A is $B \Rightarrow C$, then prove C under hypothesis B:

contexte	preuve	justification
Γ, B	`Assume` B	
Γ, B	C	$\cdots P \cdots$
Γ	`Therefore` $B \Rightarrow C$	$\Rightarrow I$

Tactic 6

If A is $B \vee C$, then prove B :

contexte	preuve	justification
Γ	B	$\cdots P \cdots$
Γ	$B \vee C$	$\vee I1$

If the proof of B fails then prove C :

contexte	preuve	justification
Γ	C	$\cdots P \cdots$
Γ	$B \vee C$	$\vee I2$

If the proof of C fails, try the following tactics.

Tactic 7

If $B \wedge C$ is in the environment, then prove A starting from formulae B, C, replacing $B \wedge C$ in the environment:

contexte	preuve	justification
$\Gamma, B \wedge C$	B	$\wedge E1$
$\Gamma, B \wedge C$	C	$\wedge E2$
$\Gamma, B \wedge C$	A	$\cdots P \cdots$

Tactic 8

If $B \vee C$ is in the environment, then:

- prove A in the environment where B replaces $B \vee C$.
- prove A in the environment where C replaces $B \vee C$.

contexte	preuve	justification
$\Gamma, B \vee C, B$	Assume B	
$\Gamma, B \vee C, B$	A	$\cdots P \cdots$
$\Gamma, B \vee C$	Therefore $B \Rightarrow A$	$\Rightarrow I$
$\Gamma, B \vee C, C$	Assume C	
$\Gamma, B \vee C, C$	A	$\cdots Q \cdots$
$\Gamma, B \vee C$	Therefore $C \Rightarrow A$	$\Rightarrow I$
$\Gamma, B \vee C$	A	$\vee E$

Tactic 9

If $\neg(B \vee C)$ is in the environment, then

- we derive $\neg B$ by the proof $P4$ and
- $\neg C$ by the proof $P5$ (proofs requested in exercise 59).
- Let P the proof of A in the environment where $\neg B$, $\neg C$ replace the formula $\neg(B \vee C)$.

The proof of A is "$P4$, $P5$, P".

Tactic 10

If A is $B \vee C$, then prove C under hypothesis $\neg B$: let P the obtained proof.
"`Assume` $\neg B$, P, `Therefore` $\neg B \Rightarrow C$" is a proof of the formula $\neg B \Rightarrow C$ which is equivalent to A.

To obtain the proof of A, simply add the proof $P1$, requested in exercise 59 of A in the environment $\neg B \Rightarrow C$.
The proof obtained from A is therefore "`Assume` $\neg B$, P, `Therefore` $\neg B \Rightarrow C$, $P1$".

Tactic 11

If $\neg(B \wedge C)$ is in the environment, then we deduce from it $\neg B \vee \neg C$ by the proof $P3$ requested in exercise 59 then we reason case by case as follows:

- prove A in the environment where $\neg B$ replaces $\neg(B \wedge C)$: Let P the obtained proof,
- prove A in the environment where $\neg C$ replaces $\neg(B \wedge C)$: Let Q the obtained proof.

The proof of A is “$P3$, `Assume` $\neg B$, P, `Therefore` $\neg B \Rightarrow A$, `Assume` $\neg C$, Q, `Therefore` $\neg C \Rightarrow A$, A”.

Tactique 12

If $\neg(B \Rightarrow C)$ is in the environment, then

- we derive B by the proof $P6$,
- $\neg C$ by the proof $P7$ (proofs requested in exercise 59).
- Let P the proof of A in the environment where B, $\neg C$ replace the formula $\neg(B \Rightarrow C)$.

The proof of A is "$P6$, $P7$, P".

Tactic 13

If $B \Rightarrow C$ is in the environment and if $C \neq \bot$, i.e. if $B \Rightarrow C$ is not $\neg B$, then,
we derive $\neg B \vee C$ in the environment $B \Rightarrow C$ by proof $P2$ from exercise 59, then we reason by cases:

- prove A in the environment where $\neg B$ replaces $B \Rightarrow C$: Let P the obtained proof,
- prove A in the environment where C replaces $B \Rightarrow C$: Let Q the obtained proof.

The proof of A is "$P2$, `Assume` $\neg B$, P, `Therefore` $\neg B \Rightarrow A$, `Assume` C, Q, `Therefore` $C \Rightarrow A$, A".

Example

Proof of Peirce's formula:

$$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$$

Proof plan

Tactic 5 is compulsory!

Proof Q:
`Assume` $(p \Rightarrow q) \Rightarrow p$
Q_1 proof or p in the environment $(p \Rightarrow q) \Rightarrow p$
`Therefore` $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

Proof Q_1 necessarily uses tactic 13 (the environment is $B \Rightarrow C = (p \Rightarrow q) \Rightarrow p$).

Hence we have to prove p both:

- in the environment $\neg B = \neg(p \Rightarrow q)$
- in the environment $C = p$.

Plan of the proof of Q_1

Proof Q_1 :

Q_{11} is the proof of $\neg B \vee C$ in the environment $B \Rightarrow C$, see exercise 59

`Assume` $\neg(p \Rightarrow q)$

Q_{12} proof of p in the environment $\neg(p \Rightarrow q)$

`Therefore` $\neg(p \Rightarrow q) \Rightarrow p$

`Assume` p

Q_{13} proof of p in the environment p

`Therefore` $p \Rightarrow p$

p

Proof of Q_1

Q_{13} is the empty proof, since $A = C = p$.

Q_{12} is the proof of $C = p$ in the environment $\neg(p \Rightarrow q)$.
Since $\neg A$ is an abbreviation of $A \Rightarrow \bot$, this proof is the proof P_6 requested in exercise 59, where $B = p$ and $C = q$.

By gluing pieces $Q_1,\ Q_{11},\ Q_{12},\ Q_{13}$, we obtain the proof Q.

The proof Q_{12} can also be done without using the tactics.

Plan

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Today

- Propositional Natural Deduction is correct and complete.
- Tactics for building a proof

Automated proofs

To automatically obtain the proofs in the system, we recommend the following software (implementing the 13 previous tactics):

http://teachinglogic.univ-grenoble-alpes.fr/DN/

People who prefer tree-like proofs can use the following software:

http://www-sop.inria.fr/marelle/Laurent.Thery/peanoware/Nd.html

Overview of the Semester

TODAY

- Propositional logic
- Propositional resolution
- Natural deduction for propositional logic *
- First order logic

MIDTERM EXAM

- Logical basis for automated proving (“first-order resolution”)
- First-order natural deduction

EXAM