Before we begin

About the midterm exam

- 2 hours
- you're allowed to bring one A4 sheet of handwritten notes
- French version available (but you should answer in English)
- Topics covered: all of propositional logic
- Typical exercices, one of them taken straight from the handout

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Schedule reminder, archives on

`https://wackb.gricad-pages.univ-grenoble-alpes.fr/inf402/`

First-order logic
Part one:
Language and Semantics of Formulae

Frédéric Prost

Université Grenoble Alpes

March 2023

Overview of the course

- Propositional logic: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$
- Interpretation: boolean functions
- Deductive systems: resolution, natural deduction
- Algorithms: Complete Strategy, DPLL, DN tactics

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- Propositional logic: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$
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- First-order logic: $\forall, \exists$
- Interpretation
- "First-order resolution"
- First-order natural deduction

Overview

Structure of first-order logic

A non-empty domain (more than two elements)

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Three categories:

- Terms representing the elements of the domain
- Relations
- Formulae describing the interactions between relations

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Two new symbols (quantifiers) in the formulae :

$\forall$ (universal quantification) and $\exists$ (existential quantification)

Examples:

- domain = members of a family
- the term $father(x)$ refers to a domain element (the father of x),
- the relation *brother* which applies to two elements,
- the formula $\forall x \exists y\ brother(y, x)$ means “everyone has a brother”.

Syllogism

Every man is mortal.
Socrates is a man.
Hence Socrates is mortal.

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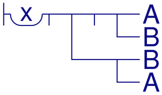
Hence Socrates is mortal.

$$\forall x(man(x) \Rightarrow mortal(x))$$
$$man(Socrates)$$
$$mortal(Socrates)$$

Gottlob Frege's *Begriffsschrift* (ideography), 1879

- Like Leibniz, attempt at a formal "universal" language 

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  (which contains rules such as Modus Ponens already known by Stoicists, but also new rules for the quantifiers)

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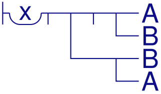
- Like Leibniz, attempt at a formal "universal" language 

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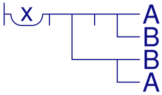

- First-order logical *system*
  (which contains rules such as Modus Ponens already known by Stoicists, but also new rules for the quantifiers)
- Contains only *reasoning* rules but allows to express every *mathematical notion* (using sets)
- Also containts **second-order** logic:
  a variable may represent a property $\forall R \exists x R(x)$

Overview

Vocabulary

- Two propositional constants: $\bot$ and $\top$
- Connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

Vocabulary

- Two propositional constants: $\bot$ and $\top$
- Connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- Quantifiers: the **universal** $\forall$ and the **existential** $\exists$
- Variables: $u, v, w, x, y, z, x1, x2\ldots$ ($\neq$ propositional vars.)
- Symbols: $a, b, c, p, brother, 12\ldots$
- Punctuation: the comma and the parentheses

Example 4.1.1

- $x, x1, x2, y$ are variables,
- *man*, *brother*, *succ*, 12, 24, $f1$, *itRains* are symbols:
  - **functions** with one, several or no arguments (constants)
  - **relations** with one, several or no arguments (propositional variables)
- For some (*special*) symbols
  we may use the infix notation $x = y$ or $z > 3$.

Term

Definition 4.1.2

A term is either :

- a symbol s alone
- or a variable
- or a symbol applied to terms $s(t_1, \ldots, t_n)$

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But $f(\perp, 2, y)$ is not a term.

Note that $42(1, y, 3)$ is also a term, but usually 42 is not used to denote a function or a relation.

Atomic formula

Definition 4.1.4 atomic formulae

An atomic formula is either:

- $\top$ or $\bot$
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Example 4.1.5:

- $P(x)$, a and $R(1, +(5, 42), g(z))$ are atomic formulae
- x and $A \vee f(4, 2, 6)$ are not atomic formulae

Beware : two-level interpretation

The set of **terms** and the set of **atomic formulae** are not disjoint.

For example $p(x)$ is **both** a term **and** an atomic formula.

Beware : two-level interpretation

The set of **terms** and the set of **atomic formulae** are not disjoint.

For example $p(x)$ is **both** a term **and** an atomic formula.

- $[\![t]\!]$ will be the value of t seen as a term
- $[t]$ will be the value of t seen as a formula.

(Strict) formula

Definition 4.1.6

A (strict) formula is either:

- an atomic formula
- $\neg A$
  where A is a formula
- $(A \circ B)$
  where A and B are formulae and $\circ$ a connective $\vee, \wedge, \Rightarrow, \Leftrightarrow$
- $\forall x\ A$ or $\exists x\ A$
  where A is a formula and x is any variable

Example 4.1.7

- $man(x), \quad brother(son(y), mother(Alice)), \quad =(x, +(f(x), g(y)))$

  are atomic formulae, hence formulae.

- On the opposite

  $\forall x\ (man(x) \Rightarrow man(Socrate))$

  is a non-atomic formula.

(Strict) formula: Examples

Among these expressions, which ones are strict formulae:

- x

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Infix notations

Prioritized formulae: the symbols of the functions $+, -, *, /$ and the symbols of the relations $=, \neq, <, >, \leq, \geq$ are written in the usual manner.

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- $\leq(*(3, x), +(y, 5))$ is abbreviated as $3 * x \leq y + 5$
- $+(x, *(y, z))$ is abbreviated as

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Example 4.1.9

- $\leq(*(3, x), +(y, 5))$ is abbreviated as $3 * x \leq y + 5$
- $+(x, *(y, z))$ is abbreviated as $x + y * z$

Prioritized formulae

Definition 4.1.10

A prioritized formula is either:

- an atomic formula
- $\neg A$
- $A \circ B$ with a binary connective $\circ$
- $\forall x\ A$ or $\exists x\ A$
- (A)

Inverse transformation

Precedence

- Quantifiers have the same precedence as negation.
- Connectives have a lower precedence than relations.
- $=, \neq, <, \leq, >, \geq$ have a lower precedence than $+, -, *, /$

Table 4.1 summary of priorities

Decreasing precedence from top to bottom.

OPERATIONS	
$-, +$ unary	
$*, /$ binary	left associative
$+, -$ binary	left associative
RELATIONS	
$=, \neq, <, \leq, >, \geq$	
NEGATION, QUANTIFIERS	
$\neg, \forall, \exists$	
BINARY CONNECTIVES	
$\wedge$	left associative
$\vee$	left associative
$\Rightarrow$	**right** associative
$\Leftrightarrow$	left associative

Tree representation

Example 4.1.12 $\forall x P(x) \Rightarrow Q(x)$

$\forall$ has higher priority: the left-hand side operand of $\Rightarrow$ is $\forall x P(x)$.

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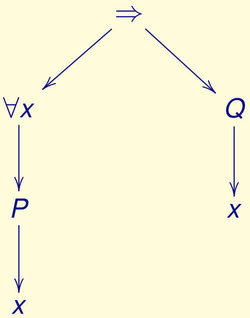

Overview

Idea

- The meaning of the formula $x + 2 = 4$ depends on x.
  The formula is not true (in arithmetics) unless $x = 2$.
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  $\forall x(x + 0 = x)$ is valid
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  There is no free variable in these two formulae.

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  x is free in the previous formula.
- $\forall x(x+2=4)$ is unsatisfiable (in arithmetics)
  $\forall x(x+0=x)$ is valid
  x does not need to be assigned a value.
  There is no free variable in these two formulae.
- Then the name of the variable doesn't matter.
  Frequent situation in mathematics $\int_0^1 f(x)dx$
  ... and in computer science

```
int Toto(int x) {
  return x + 1;
}
```

Free and bound occurrences

Definition 4.2.1

A quantifier binds a variable locally.

- In $\forall x\ A$ or $\exists x\ A$, the scope of the binding of x is A.

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If we represent a formula by a tree:

- An occurrence of x is bound if it is below a node $\exists x$ or $\forall x$.
- Any other occurrence of x is free.

Example 4.2.2

$\forall x P(\mathbf{x}, y) \land \exists z R(\underline{x}, z)$

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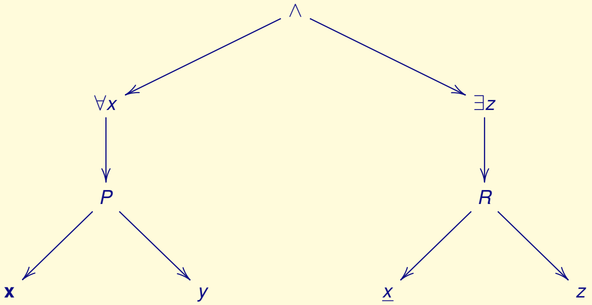
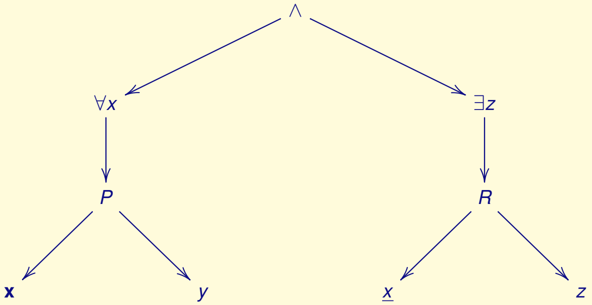


Example 4.2.2

$\forall x P(\mathbf{x}, y) \wedge \exists z R(\underline{x}, z)$



- The occurrence of z is bound, the occurrence of y is free.
- The bold occurrence of x is bound.
- The underlined occurrence of x is free.

Free, bound variables

Definition 4.2.3

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Remark

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Example 4.2.6

The free variables of $\forall x P(x, y) \wedge \exists z R(x, z)$ are x and y.

Overview

Declaring a symbol

Definition 4.3.1

A symbol declaration is a triple denoted by s^{gn} where:

- s is a symbol
- g is one of the letters f (for a function) or r (for a relation)
- n is a natural number.

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Remark 4.3.3

If the context is clear, we omit g and n.

Example: **equal** is always a 2 arguments relation.
Thus, we abbreviate the declaration $=^{r2}$ as $=$.

Symbol declaration: Example

Example 4.3.2

- $brother^{r2}$ is a **(r)elation** with **2** arguments
- $*^{f2}$ is a **(f)unction** with **2** arguments
- man^{r1} is a unary **relation**

Signature

Definition 4.3.4

A signature Σ is a set of symbol declarations.

Depending on its declaration, a symbol s will be called:

1. for s^{fn} : a function symbol with n arguments
2. for s^{f0} : a constant
3. for s^{rn} : a relation symbol with n arguments
4. for s^{r0} : a propositional variable

Example in mathematics

Let us define a signature for arithmetic:

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- Constants

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Let us define a signature for arithmetic:

- Constants $0^{f0}, 1^{f0}$
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Remarques :

- The context being well-known, we write 0, 1, $+$, $-$, $*$ and $=$.
- But note that $-$ requires two arguments
  (the symbol will not be used with only one argument).

Example in mathematics

Let us define a signature for arithmetic:

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Remarques :

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- But note that $-$ requires two arguments
  (the symbol will not be used with only one argument).

Unary relation : a relation with only 1 argument denotes a **property** of a term (for instance here $prime^{r1}$).

Term over a signature

Definition 4.3.8

A term over Σ is either:

- a variable,
- or a constant s^{f0},
- or a term $s(t_1, \ldots, t_n)$ where
  - s^{fn}
  - $n \geq 1$
  - $t_1, \ldots, t_n$ are terms over Σ.

Atomic formula over a signature

Definition 4.3.9

An atomic formula over Σ is either:

- a constant $\top$ or $\bot$
- or a propositional variable s^{r0}
- or an expression $s(t_1, \ldots, t_n)$ where
  - s^{rn}
  - $n \geq 1$
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Formula over a signature

Definition 4.3.10

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$\forall x\ (p(x) \Rightarrow \exists y\ q(x, y))$ is a formula over $\Sigma = \{p^{r1}, q^{r2}, h^{f1}, c^{f0}\}$.

But it is also a formula over the signature $\Sigma' = \{p^{r1}, q^{r2}\}$, since the symbols h and c are not in the formula.

The signature associated to a formula is the smallest signature Σ such that the formula is correctly built.

Overview

Interpretation

Definition 4.3.16

An interpretation I over a signature Σ is defined by:

- a non-empty domain D

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(function)	s_I^{fn} is a function from $D^n \to D$
(propositional variable)	s_I^{r0} is either 0 or 1
(relation)	s_I^{rn} is a set of n-uples in D
	(the ones that satisfy this relation)

Example 4.3.17

Let *friend* be a binary relation and the domain $D = \{1, 2, 3\}$.
We consider the interpretation I where
$friend_I^{r2} = \{(1,2),(1,3),(2,3)\}$.

In this interpretation, $friend(2,3)$ is true.
On the other hand, $friend(2,1)$ is false.

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Remark 4.3.18

In all interpretations, the symbol $=$ maps to the set $\{(d, d) \mid d \in D\}$.
In other words, the equality is always interpreted as the identity over D.

State, assignment

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Definition 4.3.22

An assignment is a pair (I, e) composed of an interpretation I and a state e.

Example 4.3.23

Let the domain $D = \{1,2,3\}$ and the interpretation I where $friend_I^{r2} = \{(1,2),(1,3),(2,3)\}$

The interpretation I alone does not give us the truth value of $friend(x,y)$.

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Let the domain $D = \{1,2,3\}$ and the interpretation I where $friend_I^{r2} = \{(1,2),(1,3),(2,3)\}$

The interpretation I alone does not give us the truth value of $friend(x,y)$.

Let e be the state which maps x to 2 and y to 1.

The assignment (I,e) makes the formula $friend(x,y)$ false.

Example 4.3.31

Let I be the interpretation of domain $D = \{1,2,3\}$ where $friend_I^{r2} = \{(1,2),(1,3),(2,3)\}$.

How to interpret the formula $friend(1,2) \wedge friend(2,3) \Rightarrow friend(1,3)$ in I ?

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We know how to interpret the atomic formulae:

- $[friend(1,2)]_I = true$
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We know how to interpret the atomic formulae:

- $[friend(1,2)]_I = true$
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Then we proceed as usual with the connectives, hence $[friend(1,2) \wedge friend(2,3) \Rightarrow friend(1,3)]_I = true$.
This formula is true in the interpretation I.

Overview

Finite model

Definition

A finite model of a closed formula is an interpretation of the formula in a *finite domain*, which makes the formula true.

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A finite model of a closed formula is an interpretation of the formula in a *finite domain*, which makes the formula true.

Remark

- The name of the elements of the domain is not important.
- Hence for a model with n elements, we'll use the domain of integers less than n.

Building a finite model

Naive idea: In order to know whether a closed formula has a model of domain $\{0, \ldots, n-1\}$, just

- **enumerate** all the possible interpretations of the associated signature of the formula
- **evaluate** the formula for these interpretations.

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This method is unusable in practice.

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   Let C be the obtained formula.
3. Look for a model of C by building a propositional assignment of the atomic formulae in C.

Expansion of a formula

Definition 4.3.39

The *n-expansion of A* consists in replacing:

- every sub-formula of A of the form $\forall xB$ with the conjunction $\bigwedge_{i<n} B < x := i >$
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$A = \exists x P(x) \wedge \exists x \neg P(x) \wedge \forall x \forall y (P(x) \wedge P(y) \Rightarrow x = y))$

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Which simplifies to $(P(0)+P(1)).(\overline{P(0)}+\overline{P(1)})$

The assignment $P(0) = \mathit{true}$, $P(1) = \mathit{false}$ is a propositional model of that, hence the interpretation I of domain $\{0, 1\}$ where $P_I = \{0\}$ is a model of A.

Software for building a finite model

MACE

- **translation** of first-order formulae in propositional formulae
- **performant algorithms to find the satisfiability** of a propositional formula (e.g., different versions of the `DPLL` algorithm)

`http://www.cs.unm.edu/~mccune/mace4`

An actual example:

`http://www.cs.unm.edu/~mccune/mace4/examples/2009-11A/mace4-misc/`

Overview

Today

- **First-order logic** uses the quantifiers ∀ et ∃
- We quantify over variables representing the elements of a domain
- The **atomic formulae** are built using function symbols and relations between the elements in the domain
- To give a truth value to a formula:
  - The symbols need to be interpreted in a domain
  - The free variables need to be evaluated refering to a state
- Method for finding **(counter-)model** by finite interpretation and expansion

Next lecture

- Interpretation of a first order formula
- Notion of model
- Important equivalences

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Homework: formalize in first-order logic

- Some people love each other.
- If two people are in love, then they're spouses.
- No one can love two distinct persons.