Towards proof automation: Herbrand's Theorem and Skolemization

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Reminder about expansion

Every man is mortal.	
Socrates is a man.	
Hence Socrates is mortal.	

- Look for a counter-model using a 1-expansion then a 2-exp.

- What can you conclude ?

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    We may interpret *Socrates* as 0 or 1, but neither yields a counter-model.
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    We may interpret *Socrates* as 0 or 1, but neither yields a counter-model.
- What can you conclude ?
  Nothing! Except that this formula is satisfiable.

Overview

Plan

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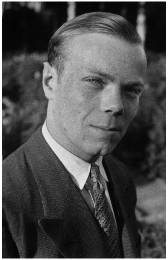
Note that if the formula is not valid, termination is not guaranteed.

Plan

Jacques Herbrand (1908-1931)

- Works in number fields
- 1930 : reduces the validity of a first-order formula to a set of propositional formulas
- Correspondence with Gödel about the consistency of arithmetic



Universal closure

Definition 5.1.1

Let C be a formula with free variables $x_1, \ldots, x_n$.

The universal closure of C, denoted by $\forall(C)$, is the formula $\forall x_1 \ldots \forall x_n C$.

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Let Γ be a set of formulae: $\forall(\Gamma) = \{ \forall(A) \mid A \in \Gamma \}$.
For example: $\forall(\{ P(x), Q(x) \}) = \{ \forall x P(x), \forall x Q(x) \}$

Assumptions

We consider that

- the formulae do not contain neither $=$, nor $\top$ or $\bot$ (since their truth value is fixed)
- every signature contains at least one constant (add an arbitrary **constant** ***a*** if need be.)

Herbrand universe (domain) and Herbrand base

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   Remark: this set is never empty, since $a \in D_{\Sigma}$.

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   $B_\Sigma = \{P(f^n(a)) \mid n \in \mathbb{N}\}$

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1. Constants symbols s are mapped to themselves.
2. If s is a function symbol and if $t_1, \ldots, t_n \in D_{\Sigma}$ then $s_{H_{\Sigma,E}}(t_1, \ldots, t_n) = s(t_1, \ldots, t_n)$.

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Another way to put it:

- Terms are interpreted as themselves.
- E is the set of true atomic formulae.

Example 5.1.8

Let $\Sigma = \{a^{f0}, b^{f0}, P^{r1}, Q^{r1}\}$

The Herbrand universe is $D_\Sigma = \{a, b\}$.

The set $E = \{P(b), Q(a)\}$ defines the Herbrand interpretation H where:

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- constants a and b are mapped to themselves and
- $P_H = \{b\}$ and $Q_H = \{a\}$.

Universal closure and Herbrand model

Theorem 5.1.16

Let Γ be a set of formulae with no quantifier over the signature Σ.

$\forall(\Gamma)$ has a model
if and only if
$\forall(\Gamma)$ has a model which is a Herbrand interpretation.

- Proof: Cf. handout course notes (just choose the "right" E)
- Consequence: no need to look for another model!

Example

Let $\Sigma = \{a^{f0}, b^{f0}, P^{r1}, Q^{r1}\}$

Let I be the interpretation of domain $\{0, 1\}$ where:

- $a_I = 0$, $b_I = 1$,
- $P_I = \{1\}$ and $Q_I = \{0\}$.

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The Herbrand universe is still $D_\Sigma = \{a, b\}$.

The set $E = \{P(b), Q(a)\}$ defines the Herbrand interpretation H where:

- Constants a and b are mapped to themselves
- $P_H = \{b\}$ and $Q_H = \{a\}$.

I is a model of a set $\forall(\Gamma)$ of formulae iff H is a Herbrand model of $\forall(\Gamma)$.

Plan

Herbrand's Theorem

Theorem 5.1.17

Let Γ be a set of formulae with no quantifiers over signature Σ.

$\forall(\Gamma)$ has a model

if and only if

Every finite set of closed instances of formulae of Γ
has a propositional model $B_\Sigma \to \{0,1\}$.

Reminders:

- Σ contains at least one constant *a* and no $=$ sign
- Instantiate = substitute each variable by a term

Other version of Herbrand's Theorem

Corollary 5.1.18

Let Γ be a set of formulae without quantifier over signature Σ.

$\forall(\Gamma)$ is unsatisfiable

if and only if

There is a finite unsatisfiable set of closed instances
of formulae taken from Γ

Proof.

Negate each side of the equivalence of the previous statement of Herbrand's theorem. □

Semi-decision procedure: unsatisfiability of $\forall(\Gamma)$

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1. if we find an unsatisfiable set, then $\forall(\Gamma)$ is unsatisfiable.
2. if we have enumerated all of them without contradiction (for a Σ *without functions*), then $\forall(\Gamma)$ is satisfiable.
3. in the meantime, we cannot conclude:
   - either $\forall(\Gamma)$ is satisfiable (and we will never stop);
   - or $\forall(\Gamma)$ is unsatisfiable but we haven't enumerated enough instances to reach a contradiction.

Example 5.1.19 (1/5)

Let $\Gamma = \{P(x), Q(x), \neg P(a) \vee \neg Q(b)\}$ and $\Sigma = \{a^{f0}, b^{f0}, P^{r1}, Q^{r1}\}$.

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$D_\Sigma = \{a, b\}$.

The set $\{P(a), Q(b), \neg P(a) \vee \neg Q(b)\}$ of instances over the D_Σ is unsatisfiable, hence $\forall(\Gamma)$ is unsatisfiable.

Example 5.1.19 (2/5)

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The set of all the instances over D_Σ is :
$\{P(a) \vee Q(a), P(b) \vee Q(b), \neg P(a), \neg Q(b)\}$
It has a propositional model characterised by $E = \{P(b), Q(a)\}$.

Hence the Herbrand interpretation associated to E is a model of $\forall(\Gamma)$.

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Remark: note that we had to consider 2 instances ($x := a$ then $x := f(a)$) of the second formula of Γ to obtain a contradiction.

Example 5.1.19 (5/5)

Let $\Gamma = \left\{ \begin{array}{l} R(x, s(x)), \\ R(x, y) \wedge R(y, z) \Rightarrow R(x, z), \\ \neg R(x, x) \end{array} \right\}$

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Indeed, $\forall(\Gamma)$ has an infinite model: the interpretation I of domain $\mathbb{N}$ with $a_I = 0$, $\quad s_I(n) = n+1$ and $R_I(x,y) = x < y$.

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Remark: $\forall(\Gamma)$ has no finite model, i.e., it is useless to look for one by n-expansions.

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Skolemization

- transforms a set of closed formulae to the universal closure of a set of formulae with no quantifier.
- preserves the existence of a model (satisfiability).

Example 5.2.1

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1. $\exists x P(x)$ is a consequence of $P(a)$
2. $P(a)$ is not a consequence of $\exists x P(x)$, but a model of $\exists x\, P(x)$ "provides" a model of $P(a)$.

   (Just choose to interpret a as an element of P_I.)

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2. C = Make B proper
3. D= Eliminate existential quantifiers from C.
   This transformation only preserves the existence of a model.
4. E = Remove the universal quantifiers from D.

E is the Skolem form of A.
(E is a normal formula with no quantifier.)

1. Normalization

1. Eliminate the equivalences
2. Eliminate the implications
3. Move the negations towards the atomic formulae

Rules

1. et 2. As in propositional logic: $\begin{cases} A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A) \\ A \Rightarrow B \equiv \neg A \vee B \end{cases}$

3. As in propositional logic: $\begin{cases} \neg\neg A \equiv A \\ \neg(A \wedge B) \equiv \neg A \vee \neg B \\ \neg(A \vee B) \equiv \neg A \wedge \neg B \end{cases}$

Furthermore $\begin{cases} \neg\forall x A \equiv \exists x \neg A \\ \neg\exists x A \equiv \forall x \neg A \end{cases}$

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First, elimination of $\Leftrightarrow$:

$$\forall y((\neg \forall x P(x,y) \vee Q(y)) \wedge (\neg Q(y) \vee \forall x P(x,y)))$$

then, move $\neg$:

$$\forall y((\exists x \neg P(x,y) \vee Q(y)) \wedge (\neg Q(y) \vee \forall x P(x,y)))$$

2. Transformation to a proper formula

Rename bound variables, e.g., by choosing new names.

Example 5.2.8

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- The formula $\forall x P(x) \vee \forall x Q(x)$ is changed to

  $\boxed{\forall x P(x) \vee \forall y Q(y)}$

- The formula $\forall x(P(x) \Rightarrow \exists x Q(x) \wedge \exists y R(x, y))$ is changed to

  $\boxed{\forall x(P(x) \Rightarrow \exists z Q(z) \wedge \exists y R(x, y))}$

Reminder: renaming of bound variables

Theorem 4.4.3

Let Q be a quantifier.
If y does not appear in $Qx\ A$ then : $\quad Qx\ A \equiv Qy\ A < x := y >$

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Example 4.4.4

- $\forall x\ p(x,z) \equiv \forall y\ p(y,z)$
- $\forall x\ p(x,z) \not\equiv \forall z\ p(z,z)$

3. Elimination of existential quantifiers

Let $\exists y B$ be a sub-formula of a closed normal and proper formula A.
Let $x_1, \dots x_n$ be the free variables of $\exists y B$.

Let f be a new symbol (if $n = 0$, then f is a constant)
and replace $\exists y B$ by $B < y := f(x_1, \dots x_n) >$ in A.

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Theorem 5.2.9

The resulting formula A' is a closed, normal and proper formula such that:

1. A is a consequence of A'
2. If A has a model then A' has an identical model (up to the truth value of f).

Remark 5.2.10

The resulting formula A' remains closed, normal and proper.

Hence, by repeatedly "applying" the theorem, choosing a **new** symbol for each eliminated quantifier, one can get:

- a closed, normal, proper formula B without $\exists$
- such that A has a model if and only if B has one.

Example 5.2.11

By eliminating existential quantifiers in the formula $\exists x \forall y P(x, y) \land \exists z \forall u \neg P(z, u)$ we obtain

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It is easy to observe that this formula has a model.

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$\forall y P(a, y) \wedge \forall u \neg P(b, u)$.
It is easy to observe that this formula has a model.

But if we mistakenly eliminate both $\exists$ using the same constant a, we obtain $\forall y P(a, y) \wedge \forall u \neg P(a, u)$

which is unsatisfiable (it entails $P(a, a)$ and $\neg P(a, a)$).

Exemple 5.2.12

By eliminating the existential quantifiers in the formula $\exists x \forall y \exists z P(x, y, z)$ we obtain

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- if we eliminate first $\exists z$:
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The existence of a model is preserved in both cases.

4. Transformation into a universal closure

Theorem 5.2.13

Let A be a closed, normal, proper formula without existential quantifier.
Let B be the formula obtained by removing all the $\forall$ from A.

A is equivalent to $\forall(B)$.

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Proof.

What we are doing is actually applying repeatedly replacements such as

- $(\forall xC) \wedge D \equiv \forall x(C \wedge D)$
- $(\forall xC) \vee D \equiv \forall x(C \vee D)$

where x is not free in D □

Property of skolemization

Property 5.2.14

Let A be a closed formula and E the Skolem form of A.
A has a model if and only if $\forall(E)$ has a model.

Proof.

A a closed formula

$\downarrow$ Normalize *(equivalent)*

B

$\downarrow$ Make proper *(equivalent)*

C

$\downarrow$ Eliminate $\exists$ *("preserves" the models)*

D

$\downarrow$ Remove $\forall$ *(equivalent to $\forall(E)$)*

E Skolem form

□

Example 5.2.15

Let $A = \forall x(P(x) \Rightarrow Q(x)) \Rightarrow (\forall x P(x) \Rightarrow \forall x Q(x))$. We skolemize $\neg A$.

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1. $\neg A$ is transformed into the normal formula:
   $\forall x(\neg P(x) \vee Q(x)) \wedge \forall x P(x) \wedge \exists x \neg Q(x)$

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   $\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \exists z \neg Q(z)$
3. The existential quantifier is "replaced" by a constant:
   $\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \neg Q(a)$

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3. The existential quantifier is "replaced" by a constant:
   $\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \neg Q(a)$
4. The universal quantifiers are removed:
   $(\neg P(x) \vee Q(x)) \wedge P(y) \wedge \neg Q(a)$.

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3. The existential quantifier is "replaced" by a constant:
   $\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \neg Q(a)$
4. The universal quantifiers are removed:
   $(\neg P(x) \vee Q(x)) \wedge P(y) \wedge \neg Q(a)$.

The instantiation $x := a, y := a$ yields $(\neg P(a) \vee Q(a)) \wedge P(a) \wedge \neg Q(a)$.

Hence (Herbrand's theorem) the Skolem form of $\neg A$ is unsatisfiable.

Since skolemization preserves satisfiability, $\neg A$ is unsatisfiable.

Plan

Today

- To prove that A is satisfiable :
  - Look for a (finite) model by n-**expansions**
- To prove that A est unsatisfiable :
  - **Skolemisation**
  - Look for a **(finite) unsatisfiable set of instances** over D_Σ
  - Herbrand's theorem: then A is unsatisfiable too
- These methods are non terminating and limited to finite interpretations
- To find a counter-model or to prove the validity of A, we proceed as before with $\neg A$

Next course

First-order deductive method:

- Clausal form
- Unification
- First-order resolution
- Consistency
- Completeness